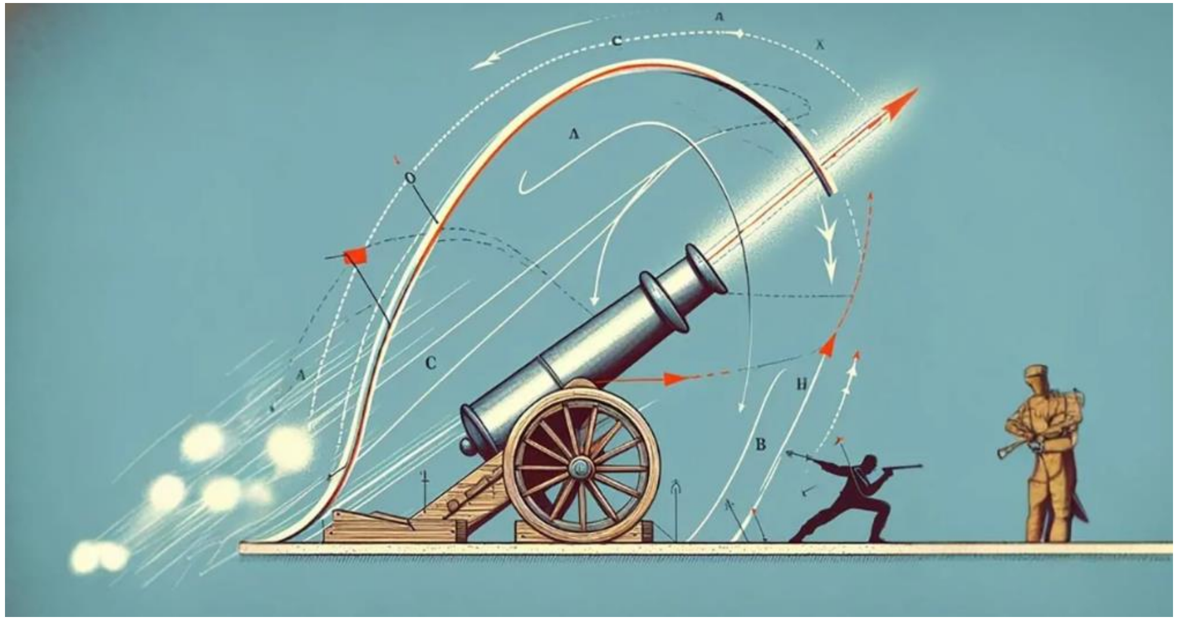
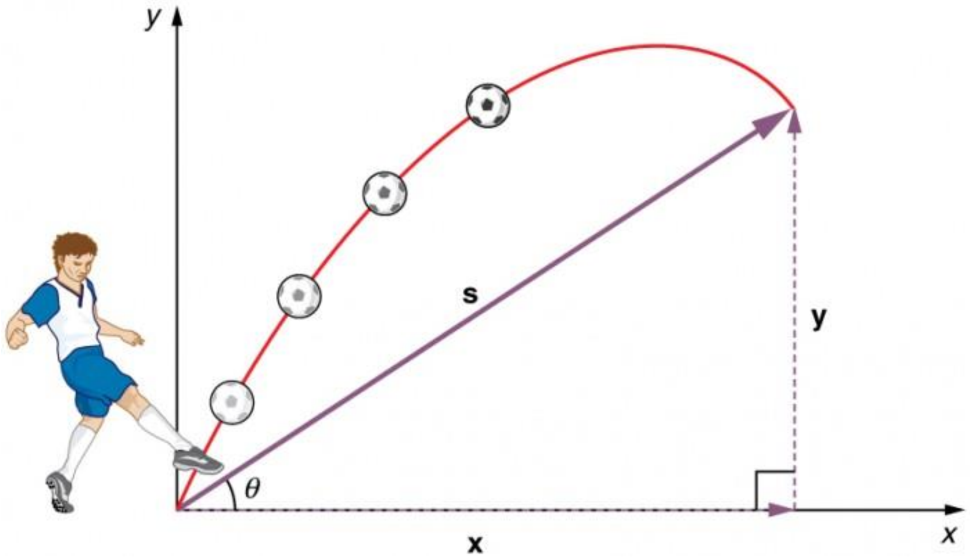




$$x(t) = V_0 t cos(\theta)$$

$$y(t) = -\frac{1}{2}gt^2 + V_0 t sin\theta$$

$$x = t^2 - t$$

$0 = t^2 - t \Rightarrow t(t-1) = 0 \begin{cases} t = 0 \\ t = 1 \end{cases}$

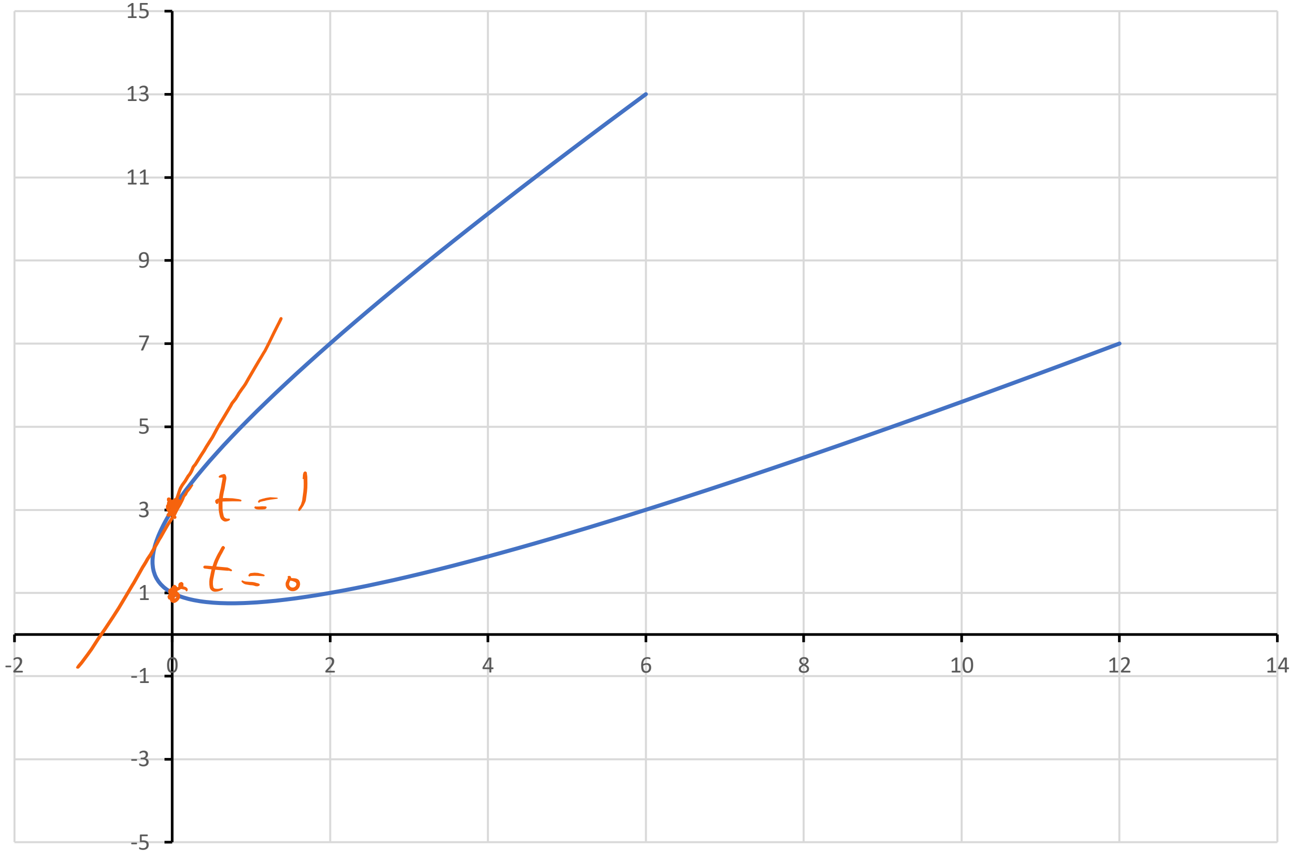
$y - y_0 = m(x - x_0)$

$$y = t^2 + t + 1$$

$$(x, y) = (0,3)$$

$m = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2t+1}{2t-1} = \dfrac{3}{1} = 3$

$\underline{y = 3x + 3}$



$x = t^2 - 2t$ $= 8$

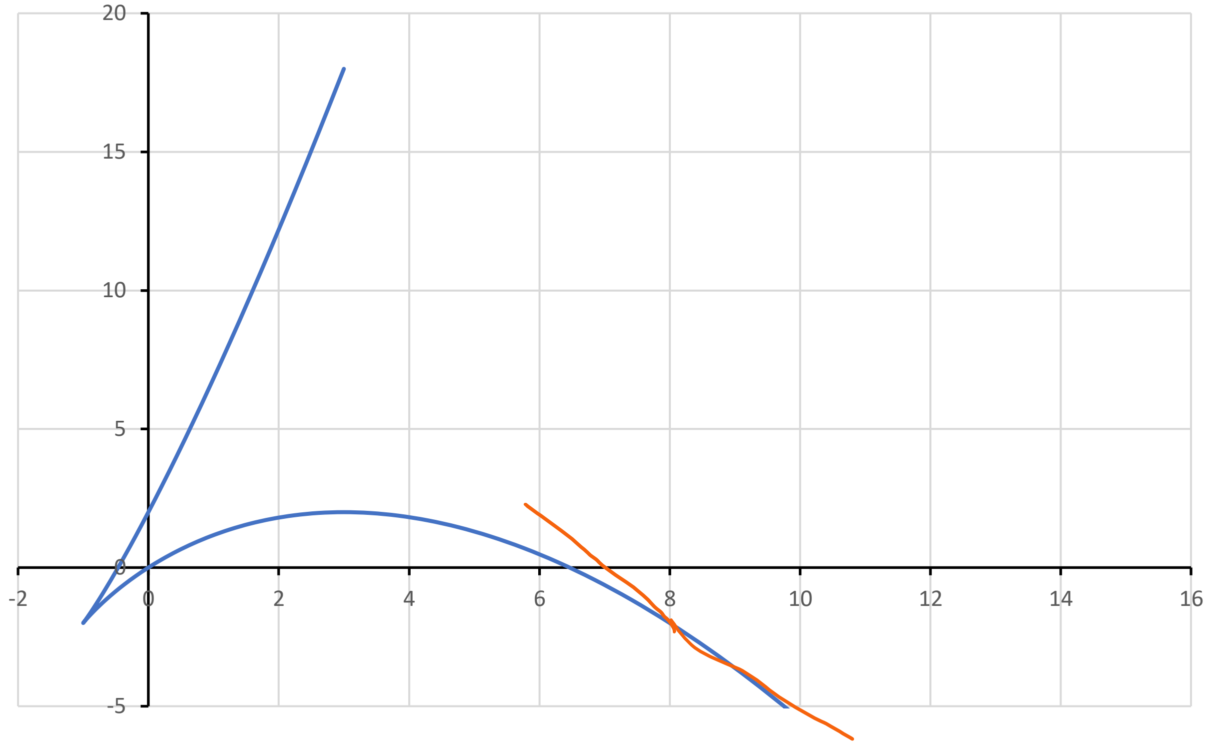
$y - (-2) = m(x - 8)$

$y = t^3 - 3t$ $-8 + 6 = -2$

$t = -2$

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t - 2} = \frac{12 - 3}{-6} = \frac{+9}{-6} = -\frac{3}{2}$$

$$y = -\frac{3}{2}(x - 8) - 2 = -\frac{3}{2}x + 10$$

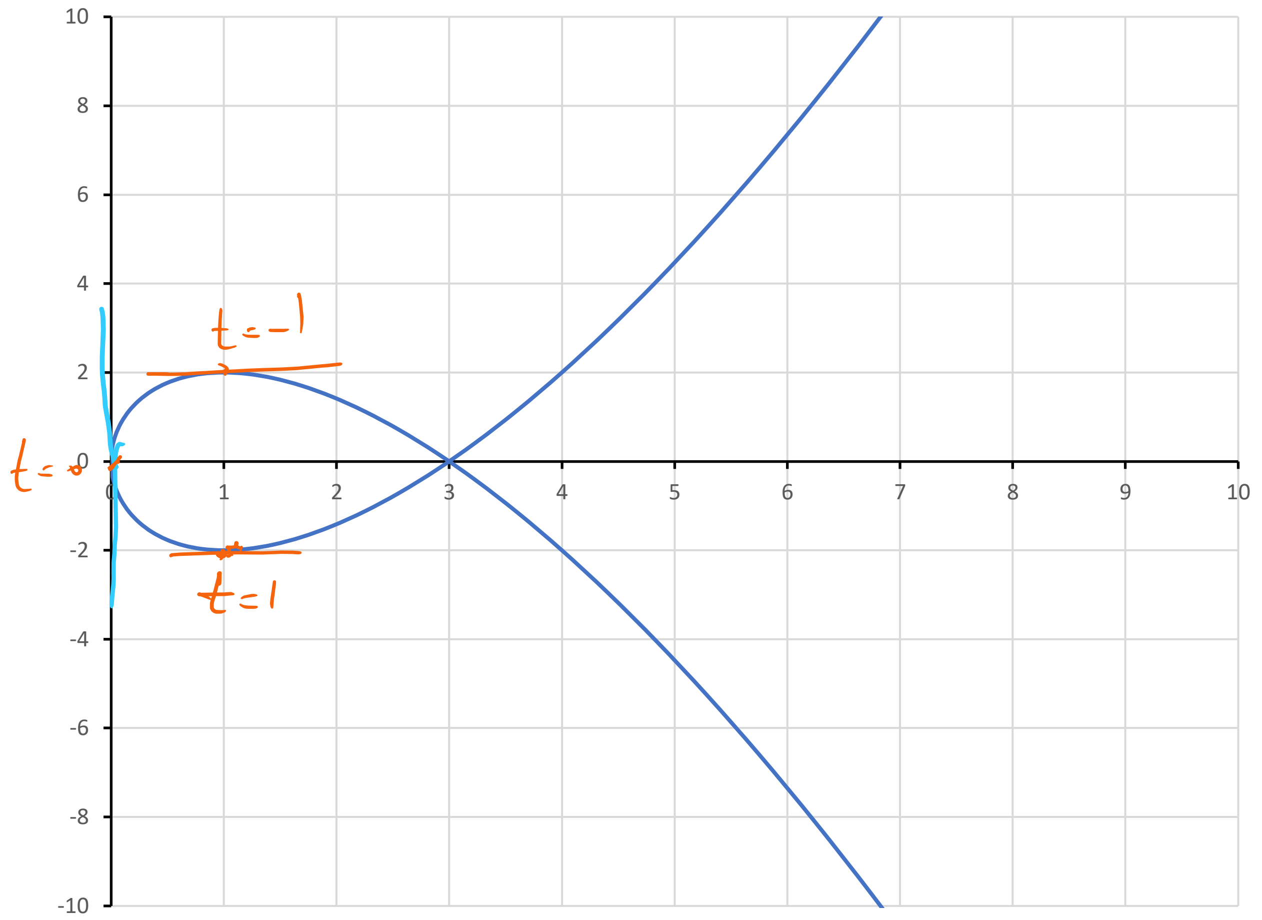


$$x = t^2$$

$$y = t^3 - 3t$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \begin{cases} \boxed{\frac{dx}{dt} = 0 \;,\; \frac{dy}{dt} \neq 0} \\ \\ \frac{dx}{dt} \neq 0 \;,\; \frac{dy}{dx} = 0 \end{cases}$$

مماس قائم



$$\frac{dy}{dt} = 3t^2 - 3 = 3(\underbrace{t^2-1}_{0}) \Rightarrow \begin{cases} t=-1 \\ t=1 \end{cases}$$

$$\frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \Rightarrow y=0 \Rightarrow t(t^2-3)=0 \begin{cases} t=0 \\ t=\pm\sqrt{3} \checkmark \end{cases}$$

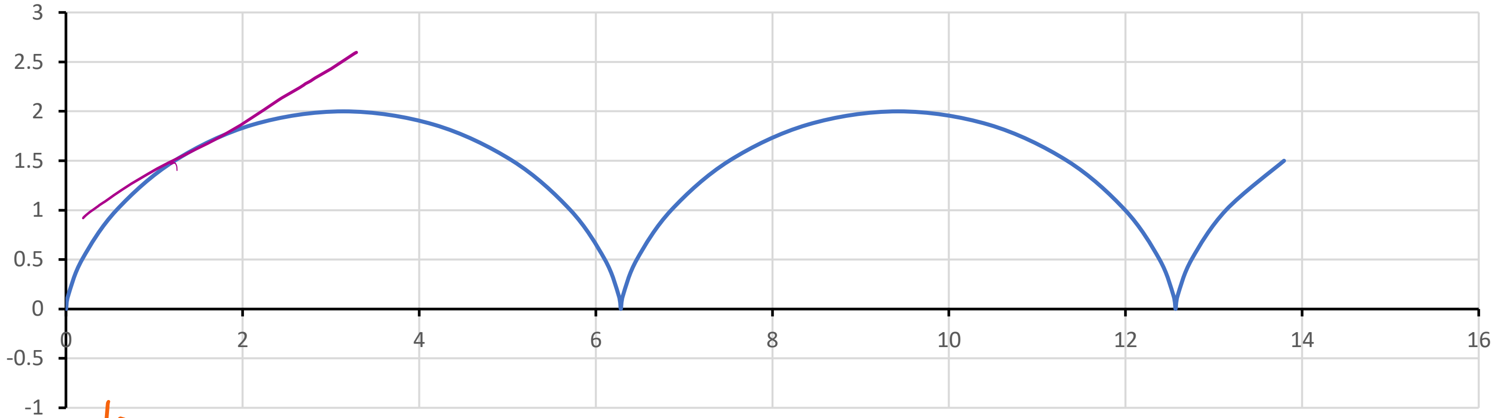
$$t=\sqrt{3} \Rightarrow \underline{x=3}, \underline{y = (\sqrt{3}^3 - 3\sqrt{3})}$$

$$m = \frac{3t^2-3}{2t} \Rightarrow \frac{6}{2\sqrt{3}} = \sqrt{3} \;,\; -\sqrt{3}$$

$$x = r(\theta - \sin(\theta))$$

$$y = r(1 - \underbrace{\cos(\theta)}_{-\sin\theta})$$

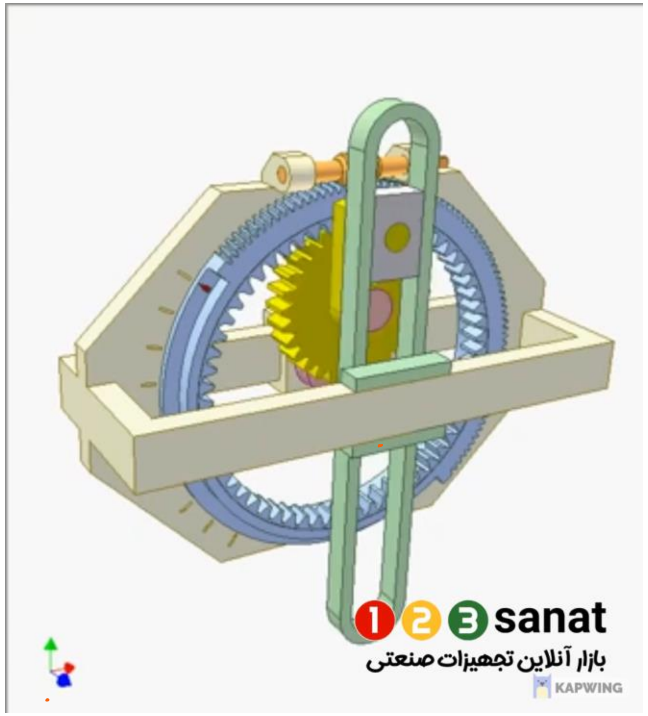
$$\theta = \frac{\pi}{3}$$



$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{r}(\sin\theta)}{\cancel{r}(1-\cos\theta)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \quad x, y$$

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \quad y = r\left(\frac{1}{2}\right) \quad \frac{r}{2}$$

$$y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

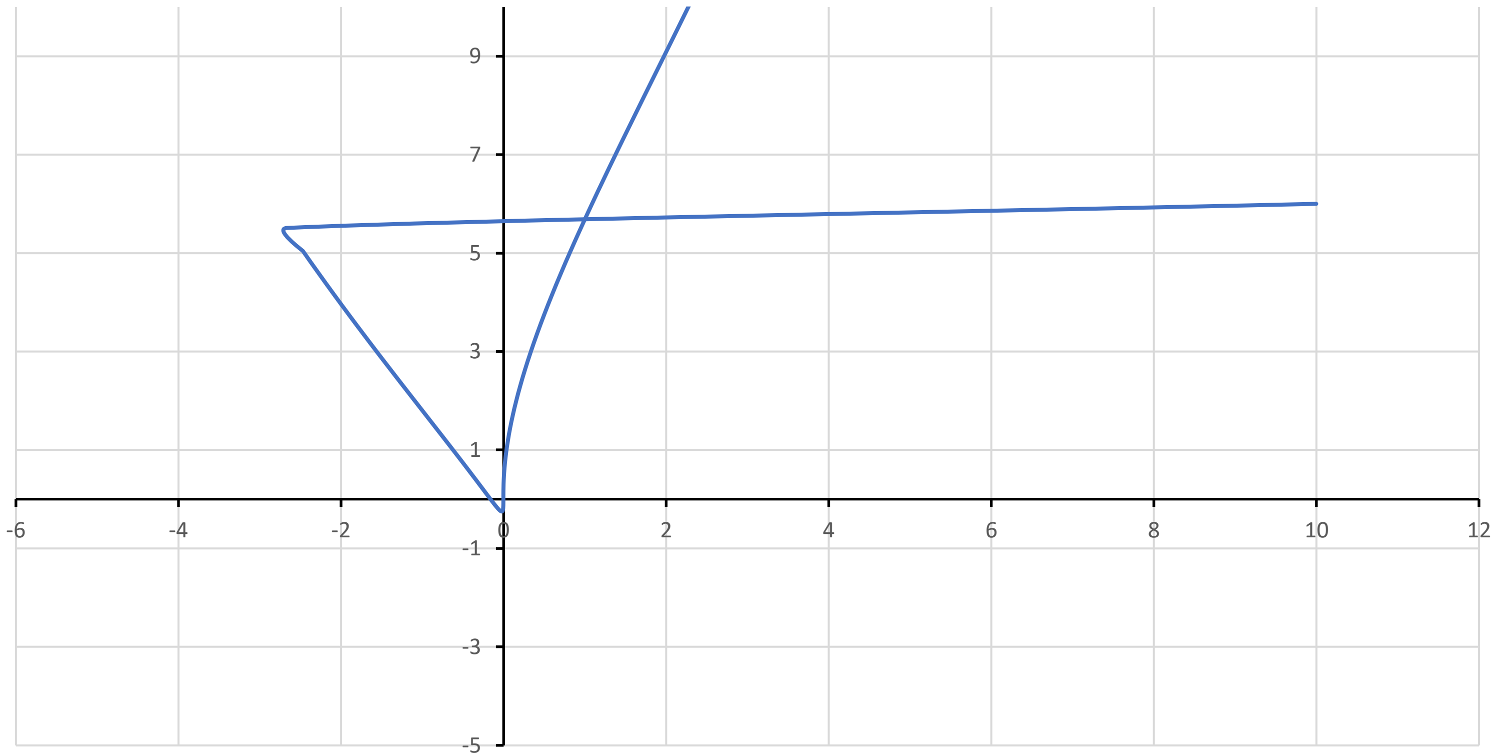


$$x = t^2 + 1$$

$$y = t^2 + t$$

$$\frac{d^2y}{dx^2} = \underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}$$

$$\underbrace{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} = \frac{2t+1}{2t}$$



$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{\overset{4t}{2(\cancel{2t})} - \overset{-4t}{2(2t+1)}}{(2t)^2}}{2t} = \frac{-2}{8t^3} = \frac{-1}{4t^3} \Rightarrow$$

تقعر -

$$x = t^2$$

$$y = t^3 - 3t$$

$$\frac{d^2y}{dx^2} =? \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dt} = 3t^2 - 3 \qquad \frac{dx}{dt} = 2t$$

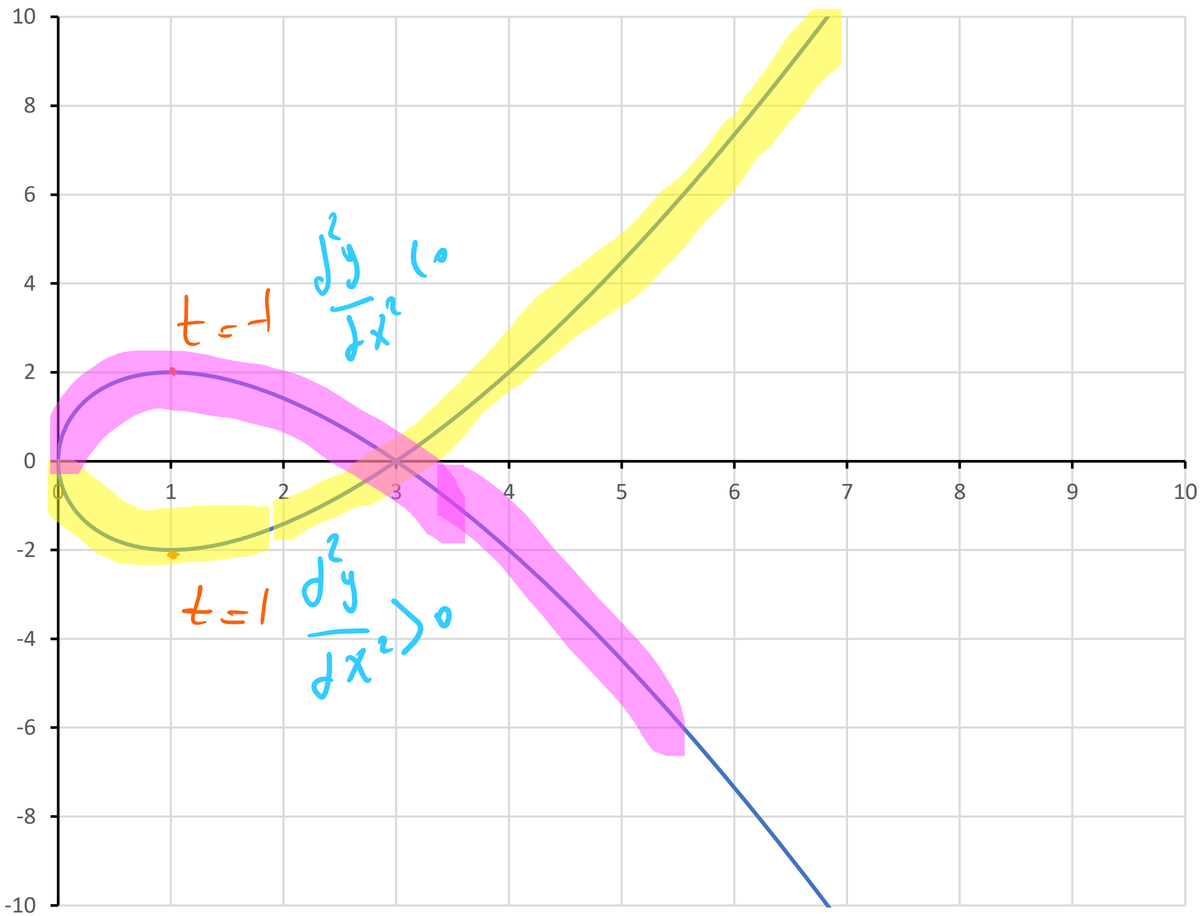
$$\frac{dy}{dx} = \frac{3t^2-3}{2t} \xrightarrow{\frac{d}{dt}} \frac{\overset{12t^2}{6t(2t)} - \overset{-6t^2+6}{\cancel{2(3t^2-3)}}}{4t^2}$$

$$= \frac{6t^2+6}{4t^2}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{6t^2+6}{4t^2}}{2t} = \frac{6(t^2+1)}{8t^3} = \frac{3(t^2+1)}{4t^3}$$

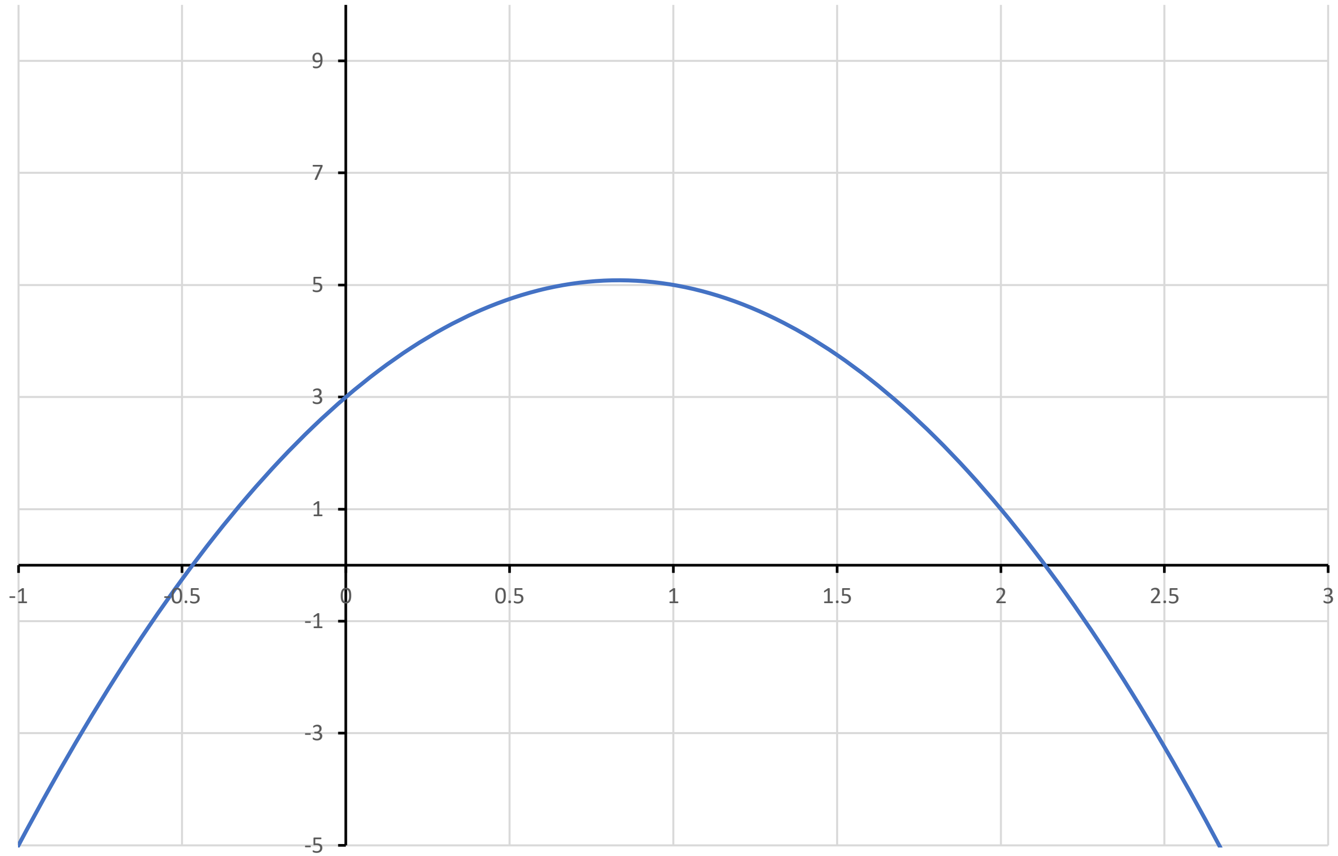
$+$



$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)dt$$

$x = f(t)$ => $dx = f'(t)dt$

$y = g(t)$



$$A = \int_a^b y\,dx = \int_\alpha^\beta \overline{g(t)}\,f'(t)\,dt$$

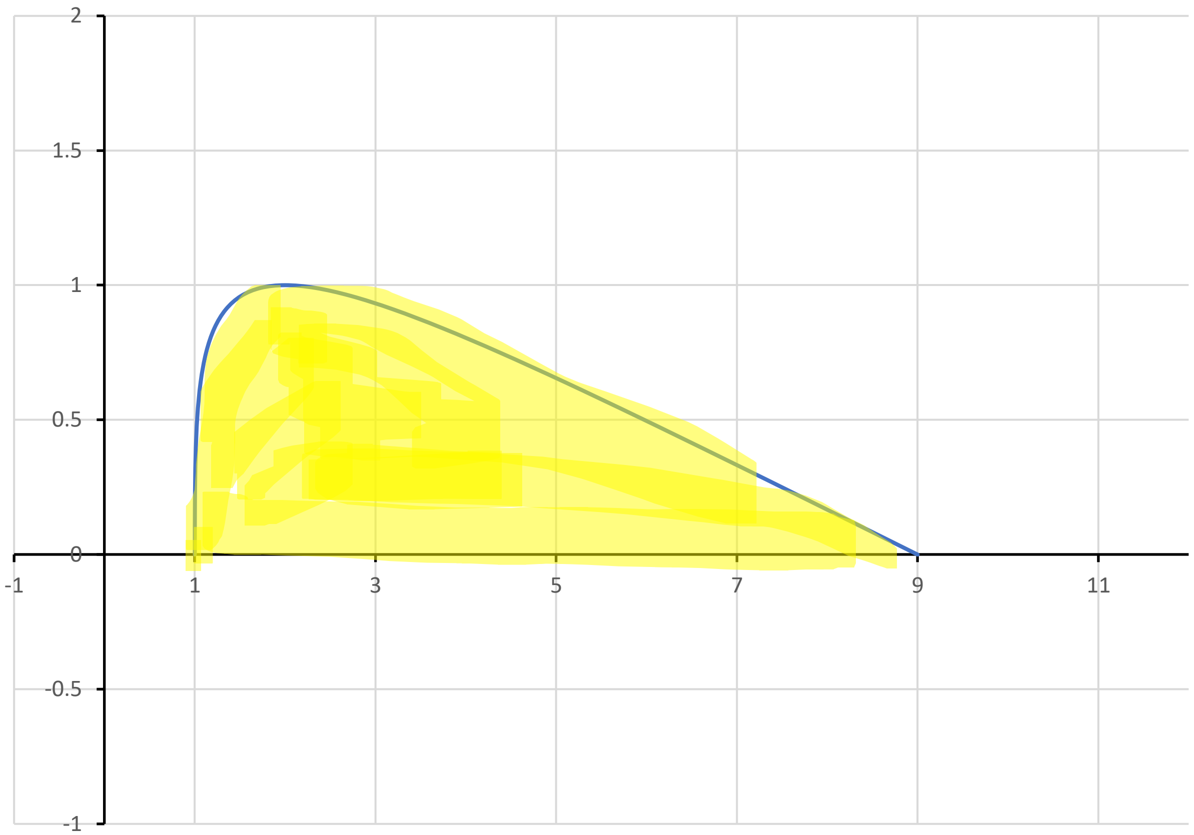
$x = t^3 + 1 \Rightarrow dx = 3t^2\,dt$

$0 < t < 2$

$y = 2t - t^2$

$$A = \int_0^2 (2t - t^2)\,3t^2\,dt = 3\int_0^2 (2t^3 - t^4)\,dt$$

$$= 3\left[\frac{2t^4}{4} - \frac{t^5}{5}\right]_0^2 = \frac{24}{5}$$



$$x = r(\theta - \sin(\theta))$$

$$dx = r(1-\cos\theta)\,d\theta$$

$$y = r(1 - \cos(\theta))$$

$$0 < \theta < 2\pi$$

$$A = \int y\,dx = \int_0^{2\pi} g(\theta)\,f'(\theta)\,d\theta$$

$$A = \int r^2(1-\cos\theta)(1-\cos\theta)\,d\theta$$

$$A = r^2\int (1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi}(1+\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)} - 2\cos\theta)\,d\theta = r^2\left[\frac{3}{2}\theta + \frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right) - 2\sin\theta\right]_0^{2\pi}$$

$$\Downarrow$$

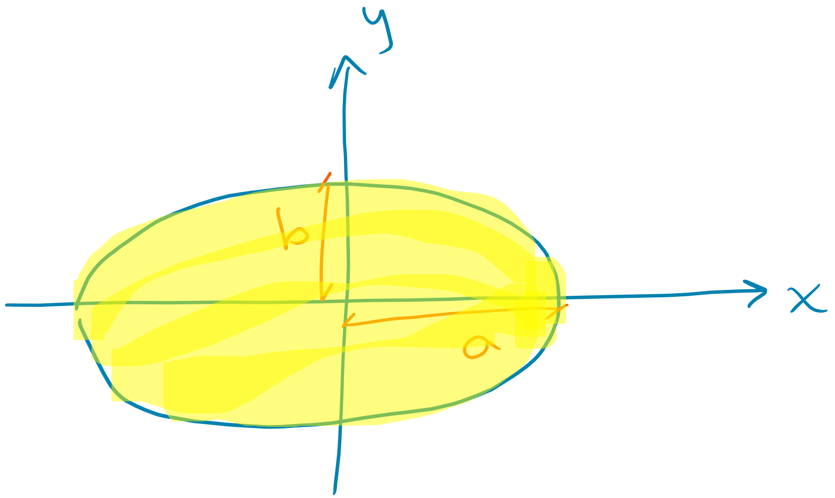
$$r^2\int_0^{2\pi}\left[\frac{3}{2} + \frac{1}{2}\cos 2\theta - 2\cos\theta\right]d\theta$$

$$= r^2\left[\frac{3}{2}(2\pi)\right] = \underline{3\pi r^2}$$

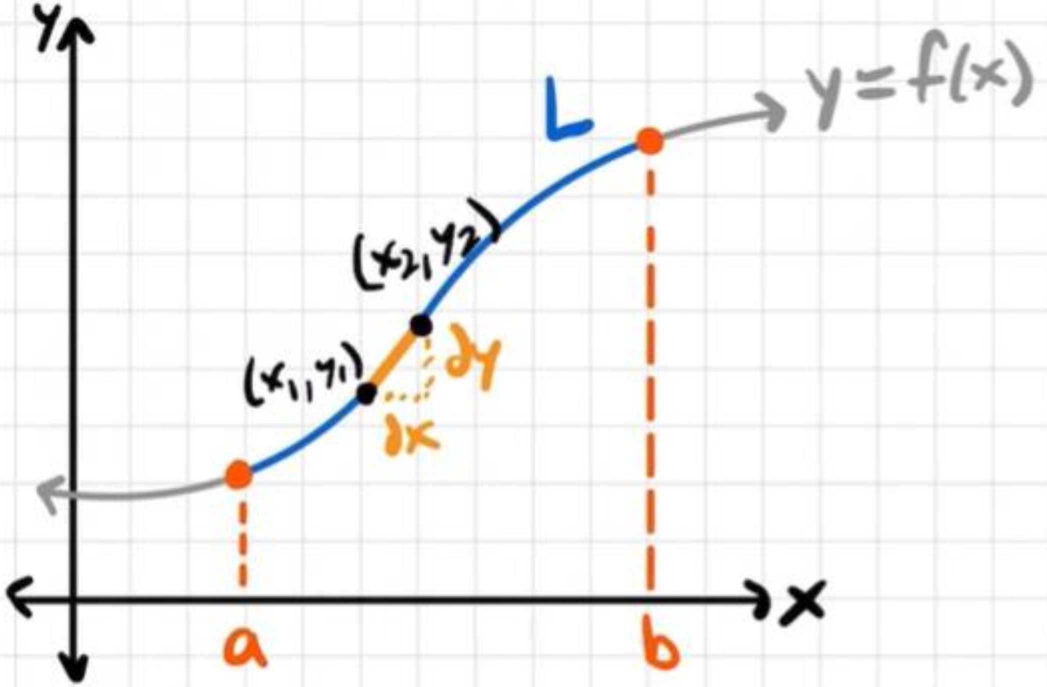
$$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\} \quad \frac{x}{a} = \cos\theta \quad \frac{y}{b} = \sin\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Lesson 6: Arc Length



$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$L=\int_a^b\sqrt{(\partial x)^2+(\partial y)^2}$$

$$=\int_a^b\sqrt{(\partial x)^2\left(1+\frac{(\partial y)^2}{(\partial x)^2}\right)}$$

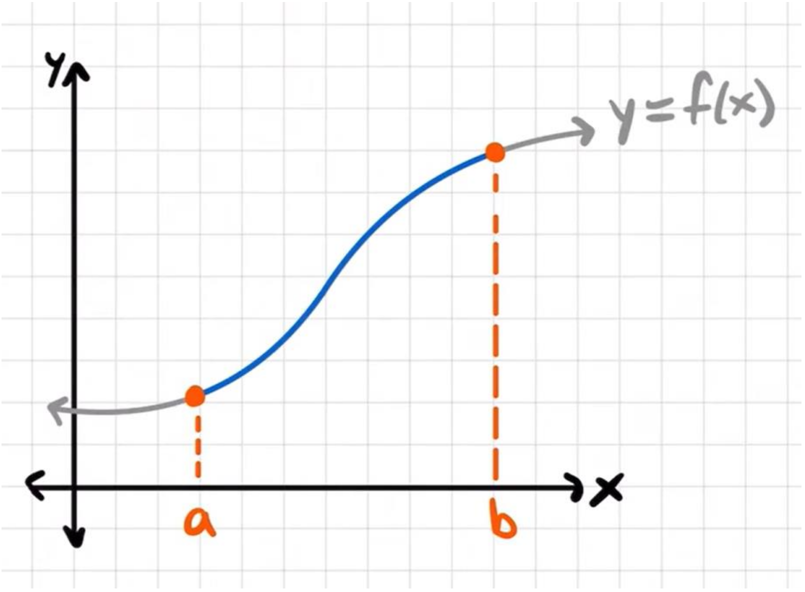
$$=\int_a^b \partial x\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}$$

$$=\int_a^b\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\partial x$$

$$\boxed{=\int_a^b\sqrt{1+(f'(x))^2}\,\partial x}$$

https://www.youtube.com/watch?v=dAXOz8ykrjQ

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

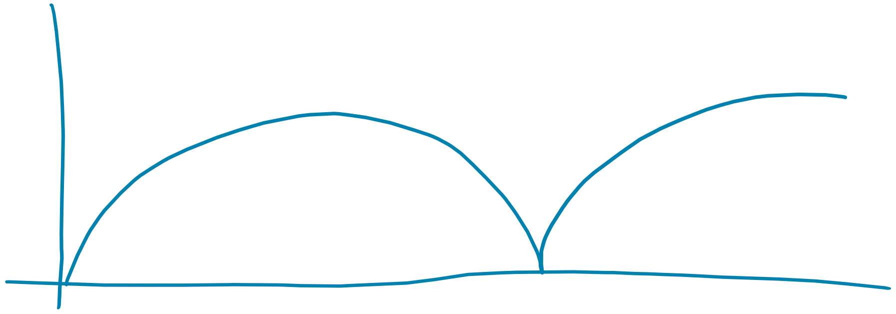


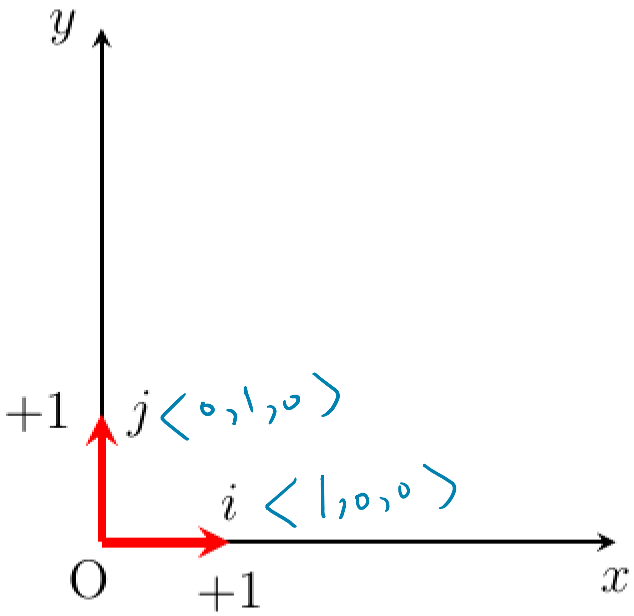
$$x = f(t)$$
$$y = g(t)$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \times \frac{dx}{dt}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

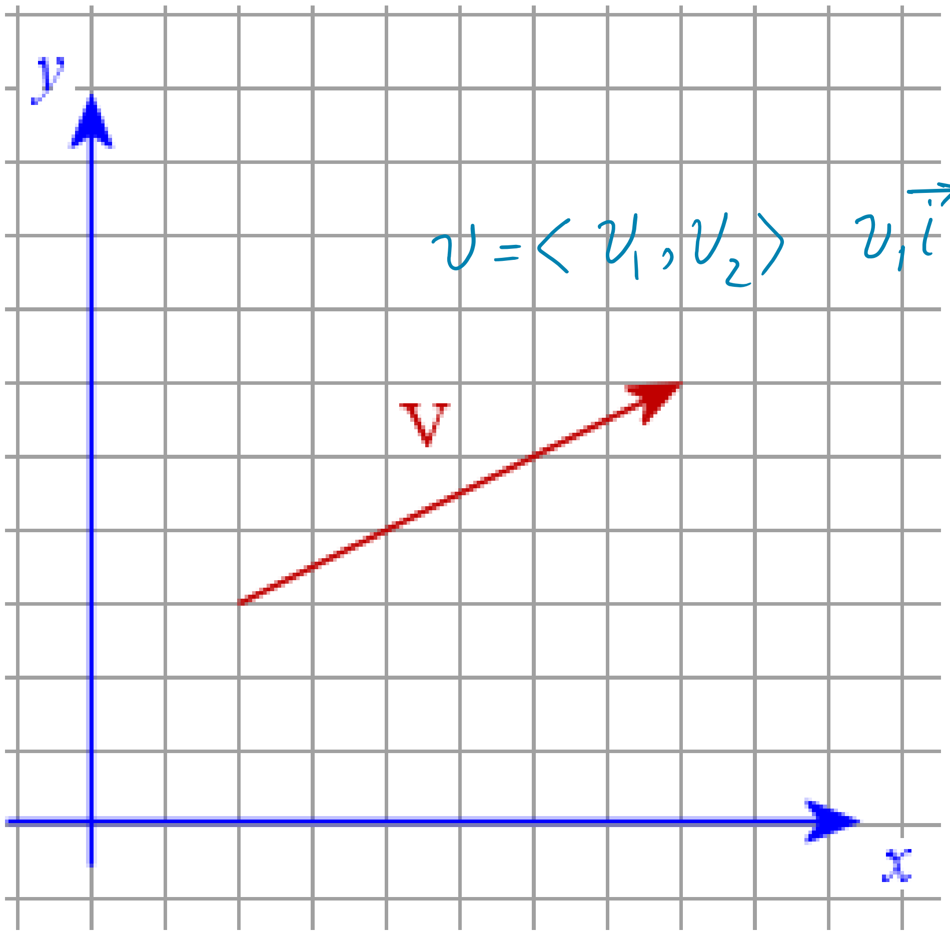
$$\begin{cases} x = r(\theta - \sin\theta) \Rightarrow \dfrac{dx}{d\theta} \\ y = r(1 - \cos\theta) \Rightarrow \dfrac{dy}{d\theta} \end{cases}$$

$\Rightarrow$ طول کمان

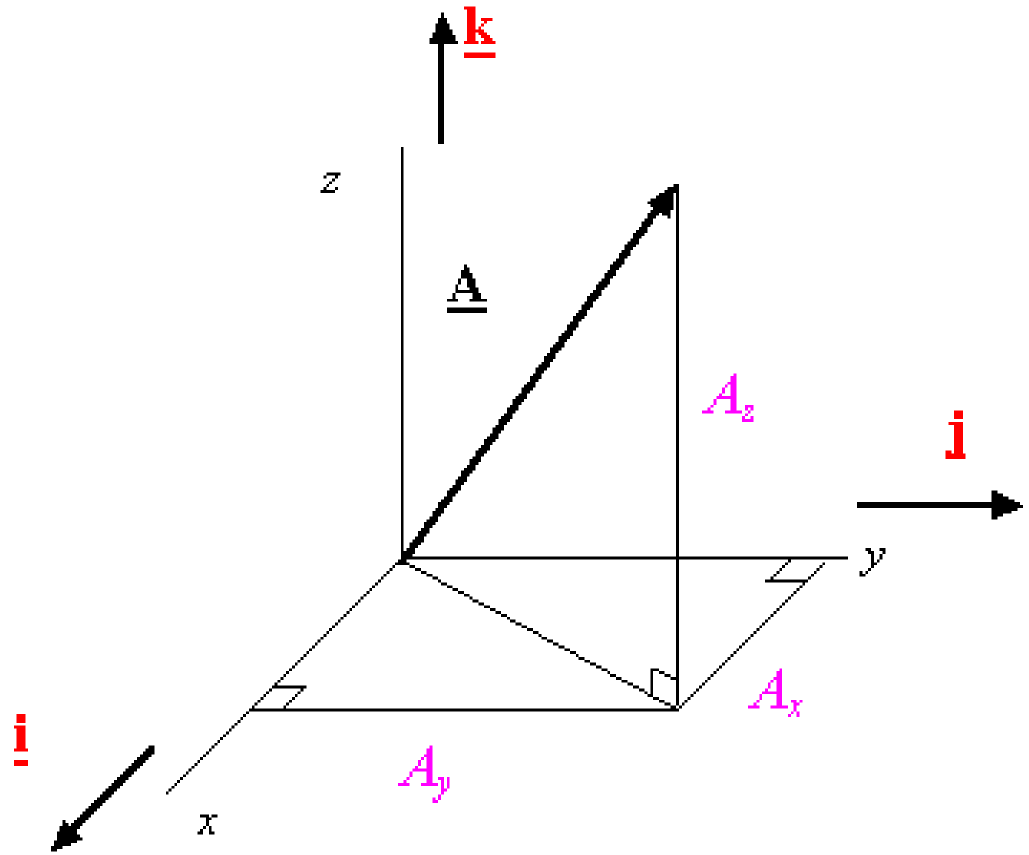




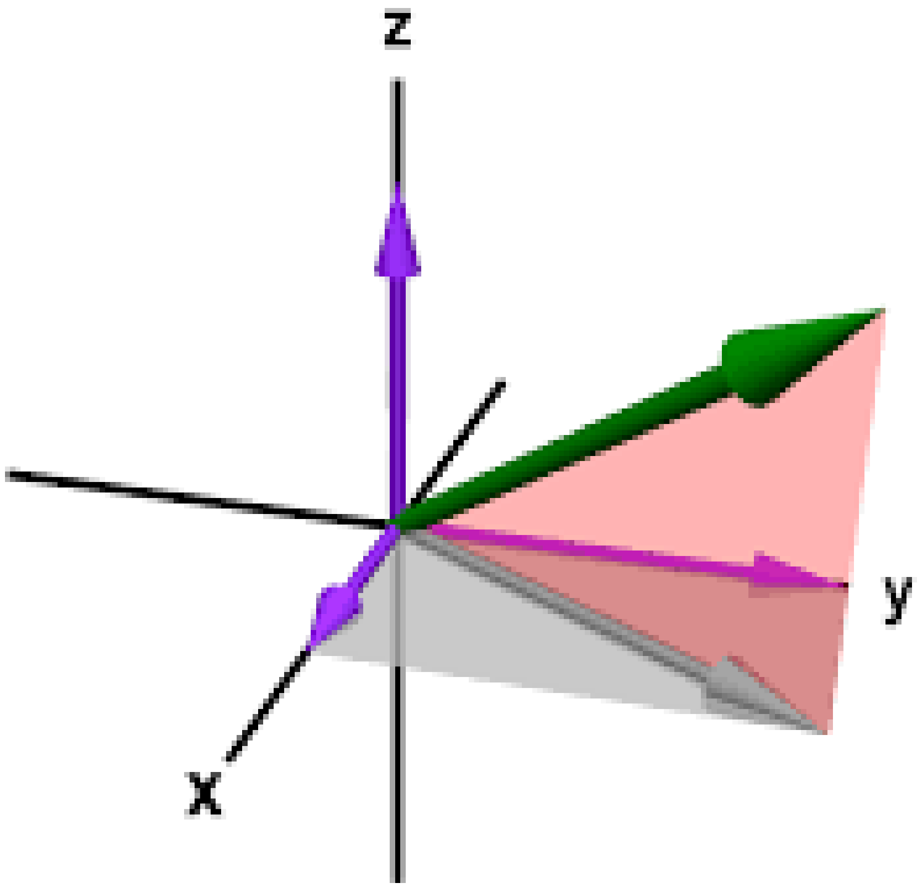
K = <0,0,1>



$v = \langle v_1, v_2 \rangle \quad v_1\vec{i} + v_2\vec{j}$



k
z
A
A_z
i
y
i
A_x
A_y
x



z
y
x

$v_1 \quad v_2$

مثال: بردار واحد هم جهت با بردار <۲− و ۶> را پیدا کنید.

$v = 6\vec{i} - 2\vec{j}$

$$\left.\begin{array}{l} v = \langle v_1, v_2, v_3 \rangle \\ |v| = \sqrt{v_1^2 + v_2^2 + v_3^2} \end{array}\right\} \frac{v}{|v|} \Rightarrow$$ بردار واحد

$$\left.\begin{array}{l} \left\langle \frac{v_1}{|v|}, \frac{v_2}{|v|} \right\rangle = \\ |v| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10} \end{array}\right\} \left\langle \frac{6}{2\sqrt{10}}, \frac{-2}{2\sqrt{10}} \right\rangle$$

$a = \langle a_1, a_2 \rangle$

$b = \langle b_1, b_2 \rangle$

$a + b = \langle a_1 + b_1, a_2 + b_2 \rangle$

$a + b = b + a$ خاصیت جابجایی دارد

$a + (b + c) = (a + b) + c$

$a + 0 = a$

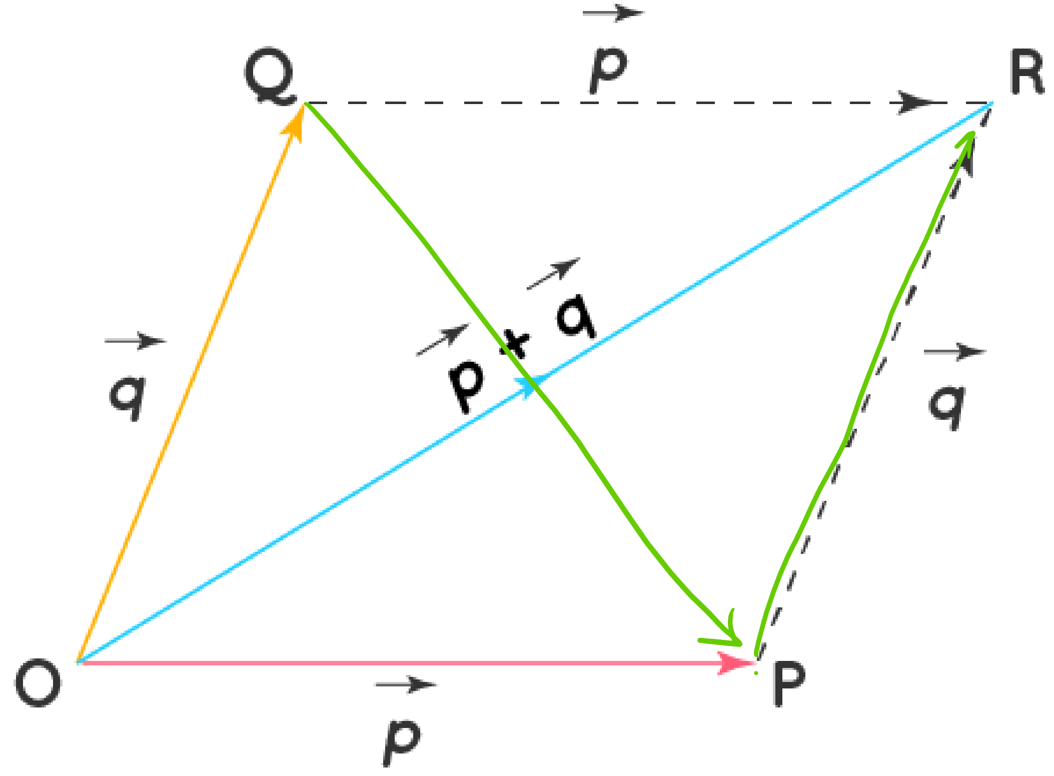
$a + (-a) = 0$

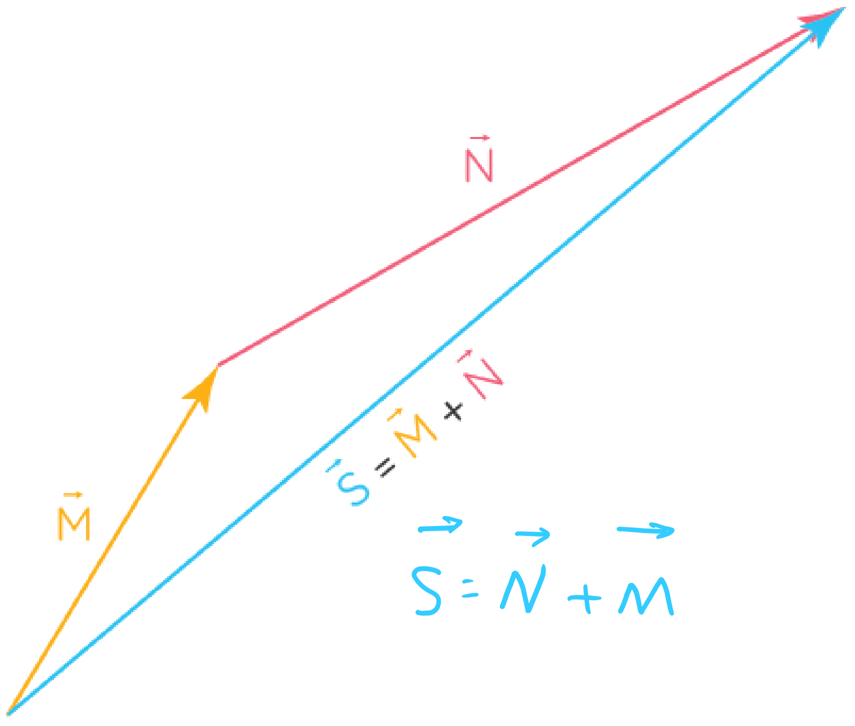
$c(a + b) = ca + cb$

$(cd)a = c(da)$

$1a = a$

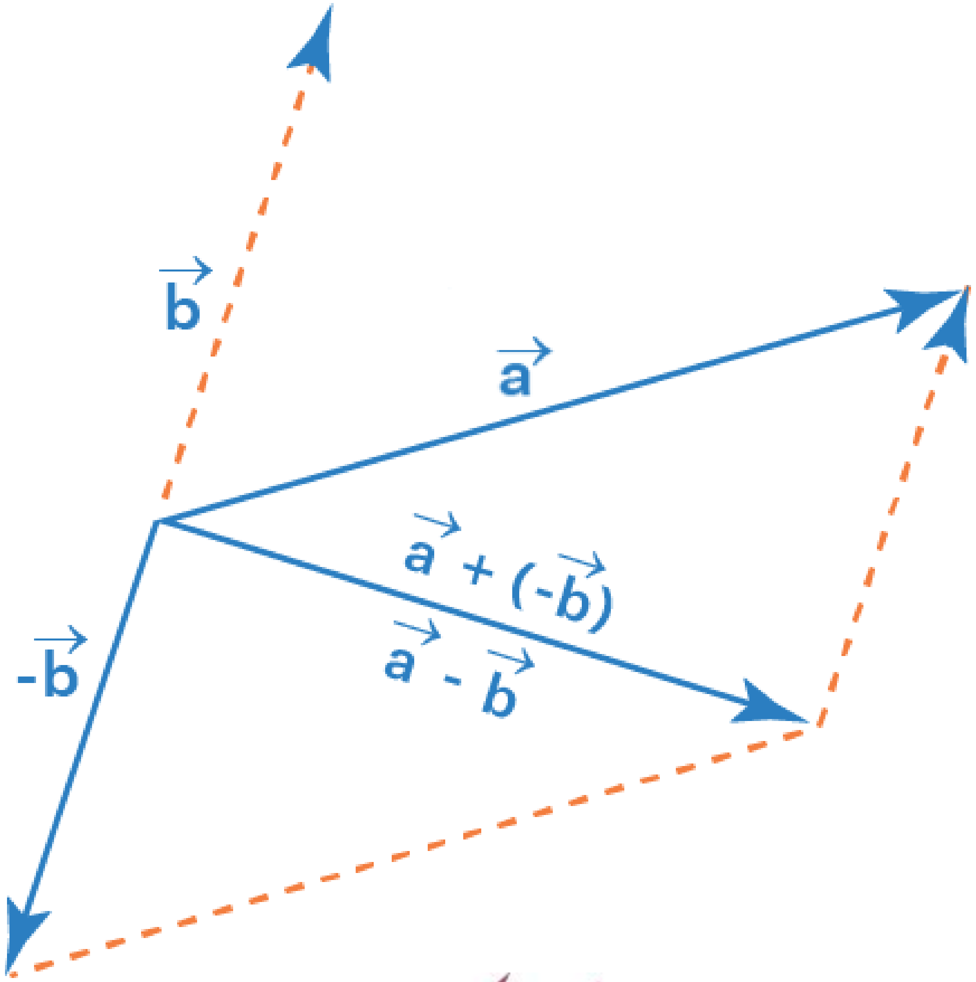
بردار { جهت / طول $\Rightarrow \vec{p}+\vec{q}=\vec{q}+\vec{p}$ خاصیت جابجایی

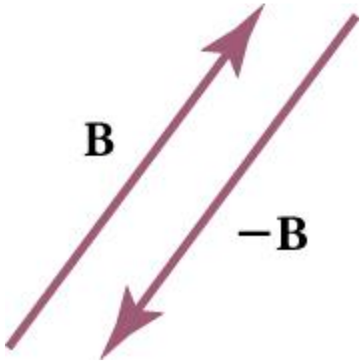


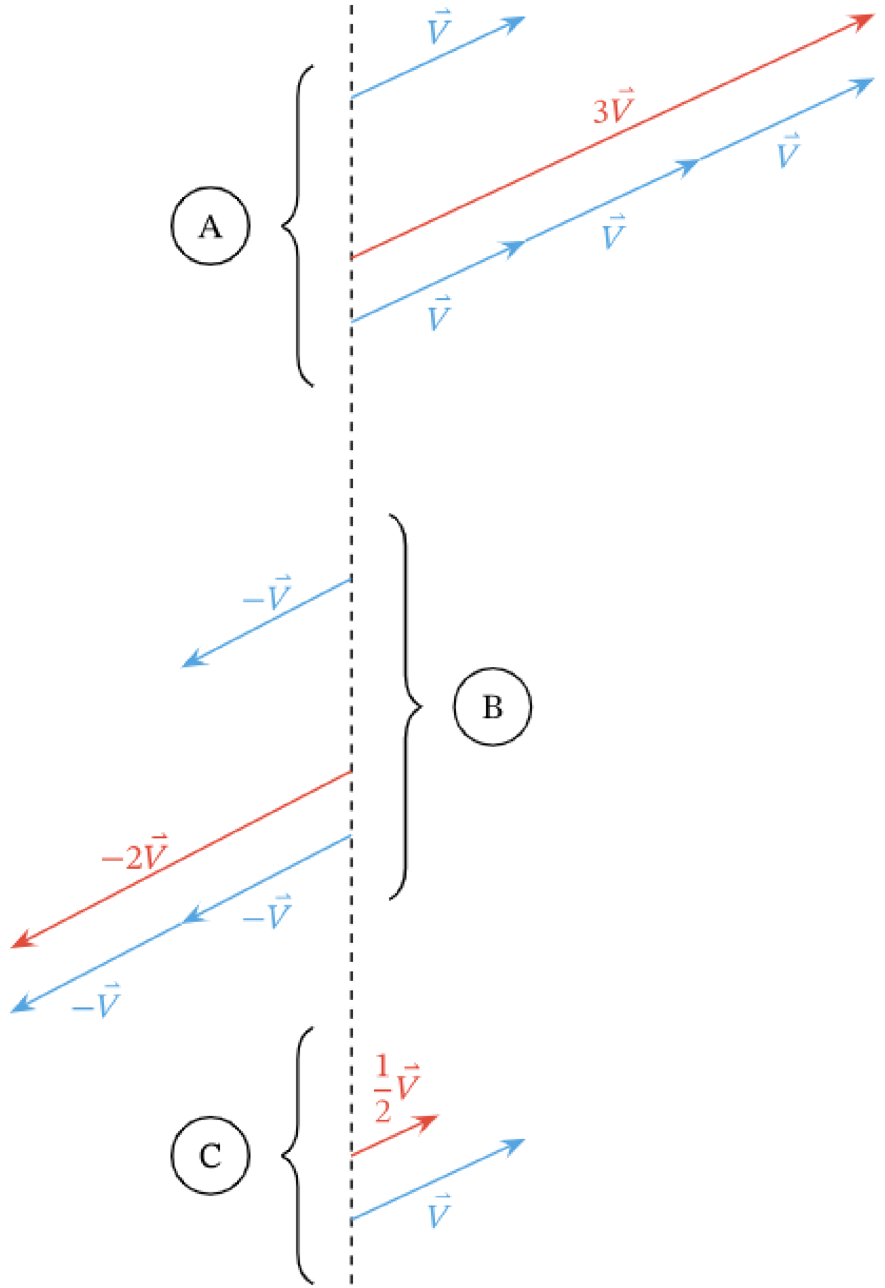


$\vec{S}=\vec{N}+\vec{M}$

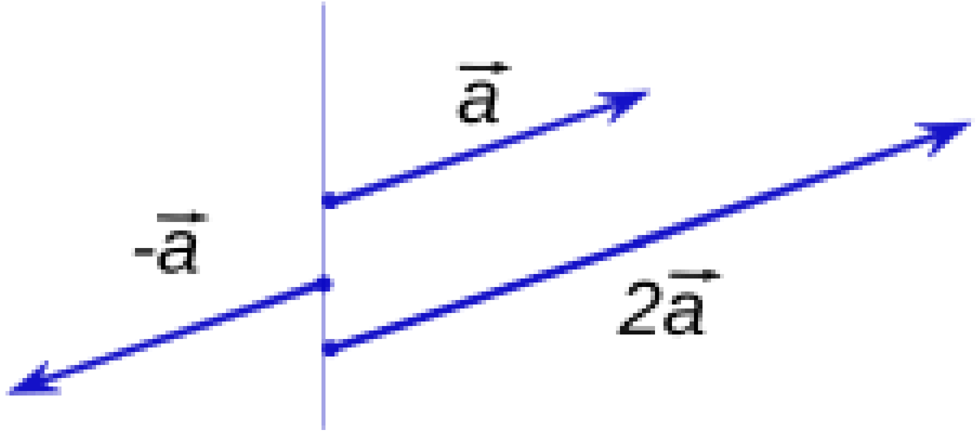
$a - b = a + (-b)$







$\vec{V}$
$3\vec{V}$
$\vec{V}$
$\vec{V}$
$\vec{V}$
A
$-\vec{V}$
B
$-2\vec{V}$
$-\vec{V}$
$-\vec{V}$
$\frac{1}{2}\vec{V}$
C
$\vec{V}$



$\vec{a}$
$-\vec{a}$
$2\vec{a}$

Dot product ضرب نقطه‌ای

$$a = \langle a_1, a_2, \cdots, a_n \rangle$$
$$b = \langle b_1, b_2, \dots, b_n \rangle$$

$$a \cdot b = \underbrace{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}_{\text{یک عدد}}$$

$$a \cdot a = |a|^2$$

$$a \cdot b = b \cdot a \quad \text{جابجایی}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$\boxed{a \cdot b = |a||b| \cos\theta}$$

$$\left.\begin{array}{l} a = \langle 2, -1, 2 \rangle \\ b = \langle 1, -1, 0 \rangle \end{array}\right\} \text{زاویه بین دو بردار} = ?$$

$$|a| = \sqrt{9} = 3$$
$$|b| = \sqrt{2}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{\overset{3}{2+1+0}}{\sqrt{2} \times 3} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4} \text{ rd} \quad 45^\circ$$

$a: 2\vec{i} + 2\vec{j} - \vec{k}$

$b: 5\vec{i} - 4\vec{j} + 2\vec{k}$

$a \cdot b = 0 \Rightarrow 10 - 8 - 2 = 0$

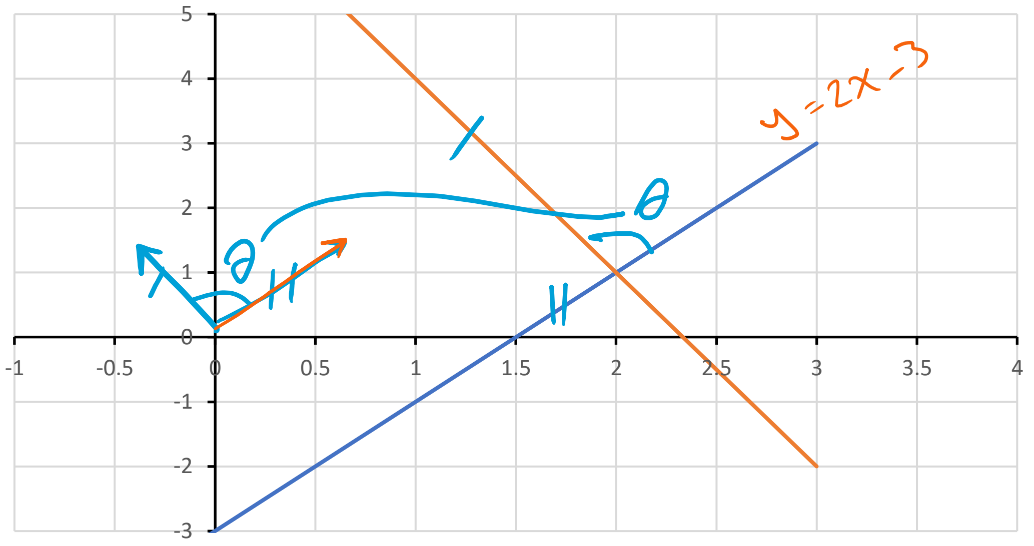
بر هم عمودند

$$\begin{cases} 2x - y = 3 \\ \quad y = 2x - 3 \\ 3x + y = 7 \end{cases}$$

$y = 7 - 3x$

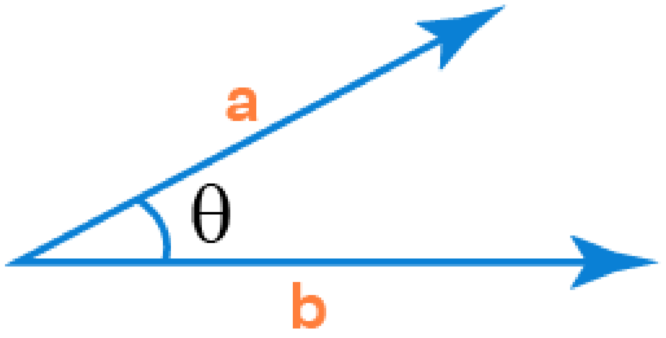
$a: \langle 1, 2 \rangle$

$b: \langle 1, -3 \rangle$



$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{1 - 6}{\sqrt{5} \times \sqrt{10}} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \begin{array}{l} \theta = 45^\circ \\ \theta = 135^\circ \end{array}$$

ویژگی‌های بردارها



$$\cos\theta = \frac{a.b}{|a|\,|b|}$$

(OR)

$$\sin\theta = \frac{|a \times b|}{|a|\,|b|}$$

https://www.cuemath.com/geometry/angle-between-vectors/

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

ماتریس همانی $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ماتریس قطری $= \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

جمع و تفریق دو ماتریس

درایه به درایه

$$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 1 & 5 \end{bmatrix}$$

ضرب عدد در ماتریس

$$2 \times \begin{bmatrix} 3 & 2 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & -18 \end{bmatrix}$$

$$\underbrace{A}_{n\times k} \times \underbrace{B}_{k\times m} = \underbrace{C}_{n\times m}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1\times 10 + 2\times 20 + 3\times 30 & 1\times 11 + 2\times 21 + 3\times 31 \\ 4\times 10 + 5\times 20 + 6\times 30 & 4\times 11 + 5\times 21 + 6\times 31 \end{bmatrix}_{2\times 2}$$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \\ 6 & 8 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & 8 \\ 5 & 6 & 7 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(A \times B)^T = B^T \times A^T \neq A^T \times B^T$$

* ماتریس متقارن

$$A = A^T$$

* اثر ماتریس

جمع مقادیر قطر اصلی $\Leftarrow \; tr(A) = \sum_{i=1}^{n} a_{ii}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad tr(A) = 1 + 5 + 9 = 15$$

* نرم ماتریس

$$\|A\|_F = \sqrt{\sum a_{ij}^2} = \sqrt{Tr(AA^T)}$$

* دترمینان ماتریس

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow det(A) = |A| = ad - bc$$

$$det \begin{bmatrix} 8 & 3 \\ 4 & 2 \end{bmatrix} = \underbrace{8\times 2}_{16} - \underbrace{3\times 4}_{12} = 4$$

$$A = \begin{bmatrix} a_+ & b_- & c_+ \\ d_- & e_+ & f_- \\ g_+ & h_- & i_+ \end{bmatrix}$$

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$sgn = (-1)^{i+j}$$

$$A = \left[\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right]\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

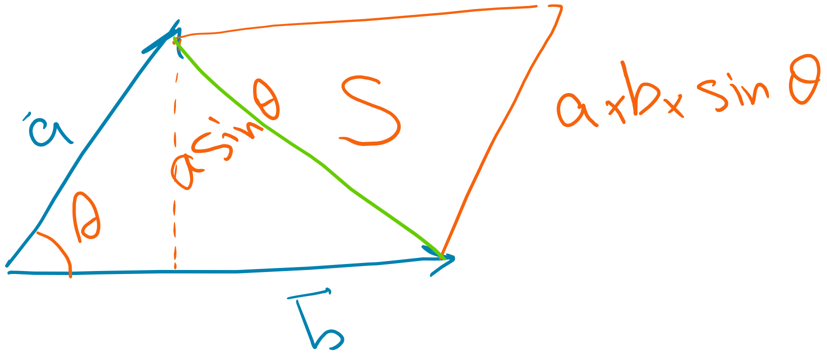
$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

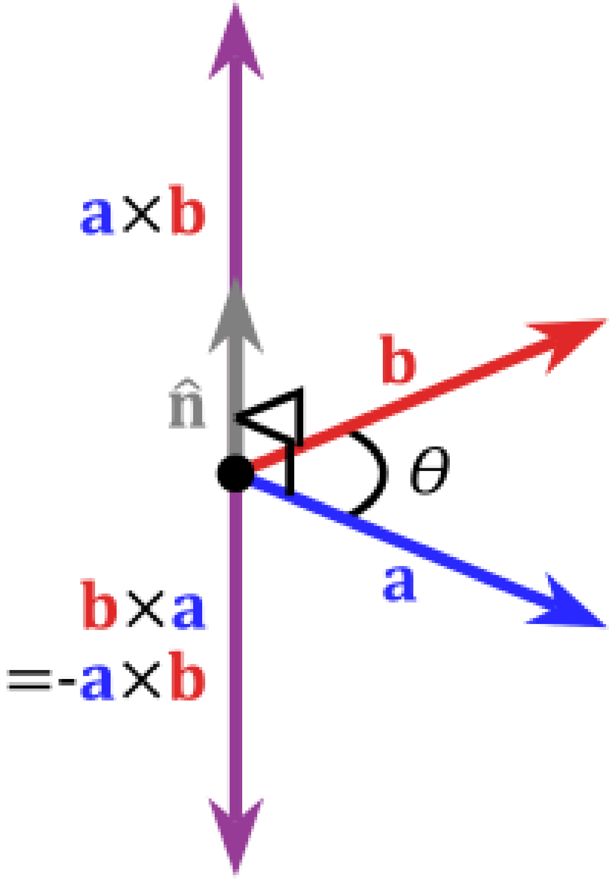
$$a = \langle a_1, a_2, a_3 \rangle \qquad a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$b = \langle b_1, b_2, b_3 \rangle \qquad b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$|a \times b| = |a||b| \sin\theta$$





$A(0,0,0)$

$B(1,0,2) \Rightarrow$ مساحت مثلث ؟

$C(2,-2,0)$

ماتریس وارون

$$A \longrightarrow A^{-1}$$

$$A \cdot A^{-1} = I$$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1} \times A^{-1}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{8-9} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

$$A^{-1}_{3\times 3} = \frac{1}{|A|} \left[\text{Adjoint} \right]$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^{T} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$(-1)^{i+j}$$

$$\operatorname{adj}(A) = \begin{bmatrix} +\begin{vmatrix} e & h \\ f & i \end{vmatrix} & -\begin{vmatrix} b & h \\ c & i \end{vmatrix} & +\begin{vmatrix} b & e \\ c & f \end{vmatrix} \\ -\begin{vmatrix} d & g \\ f & i \end{vmatrix} & +\begin{vmatrix} a & g \\ c & i \end{vmatrix} & -\begin{vmatrix} a & d \\ c & f \end{vmatrix} \\ +\begin{vmatrix} d & g \\ e & h \end{vmatrix} & -\begin{vmatrix} a & g \\ b & h \end{vmatrix} & +\begin{vmatrix} a & d \\ b & e \end{vmatrix} \end{bmatrix}$$

وارون ماتریس قطری

$$A=\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

مقادیر ویژه و بردار ویژه

مقدار ویژه

$$Ax = \lambda X$$

بردار ویژه

$$Ax - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

مقدار ویژه

بردار ویژه

$a\ (1,1)$

$a\ (a_1, a_2)$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \Rightarrow \overbrace{\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}}^{A} - \lambda \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{I} = \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

$$(1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda \begin{cases} -2 \\ 5 \Rightarrow x_1 = x_2 \Rightarrow \begin{bmatrix} a \\ a \end{bmatrix} \end{cases}$$

$$\begin{cases} -4x_1 + 4x_2 = 0 \rightarrow x_1 = x_2 \\ 3x_1 - 3x_2 = 0 \quad x_1 = x_2 \end{cases}$$

$$\begin{bmatrix} \underbrace{1-(-2)}_{3} & 4 \\ 3 & \underbrace{2-(-2)}_{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 4x_2 = 0 \Rightarrow x_1 = -\frac{4}{3}x_2 \Rightarrow \begin{bmatrix} -\frac{4}{3}a \\ a \end{bmatrix}$$

$$3x_1 + 4x_2 = 0$$

دستگاه معادلات خطی و قانون کرامر

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1$$

$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m$$

$$A \times X = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = b$$

$$\underbrace{A^{-1} \times A}_{I} \times X = A^{-1}b \Rightarrow X = A^{-1}b$$

$$\begin{cases} x_1 - x_2 = -1 \Rightarrow x_1 = -1 + x_2 \xrightarrow[\text{در معادله } 2]{\text{قرار دادن}} 3(-1+x_2) + x_2 = -1 \Rightarrow \begin{cases} x_2 = 3 \\ x_1 = 2 \end{cases} \\ 3x_1 + x_2 = 9 \end{cases}$$

$$Ax = b$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}[\operatorname{adj}(A)] = \frac{1}{\underbrace{1-(-3)}_{4}} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}^{-1}}_{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}} \begin{bmatrix} -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

روش کرامر

$$x_1 = \frac{\begin{vmatrix} -1 & -1 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{-1-(-9)}{1-(-3)} = \frac{8}{4} = 2 \qquad x_2 = \frac{\begin{vmatrix} 1 & -1 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{9-(-3)}{4} = \frac{12}{4} = 3$$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 4 & 0 & 8 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{\frac{R_1}{4}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \begin{array}{c} x_1 \\ x_2 \end{array}$$

$$\underbrace{\begin{matrix} 1 & -1 \\ 3 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} -1 \\ 9 \end{matrix}}_{\text{پاسخ}}$$

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 5 \\ 3x_1 + 5x_2 + 6x_3 = 7 \\ 2x_1 + 4x_2 + 3x_3 = 8 \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}} = -15 \qquad x_2 = -8 \qquad x_3 = 2$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix} = 1(5\times3 - 6\times4) - 3(3\times3 - 6\times2) + (-2)(3\times4 - 5\times2)$$

به ازای چه مقداری از a دستگاه زیر بینهایت جواب دارد

$$\begin{cases} x - y + az = 0 \\ 3x - y + 2z = 0 \\ 6x - 4y + 5z = 0 \end{cases}$$

$AX = 0$

$|A| = 0 \Rightarrow$ بی‌نهایت جواب

معادلات خط و صفحه در فضای سه بعدی

$$y - y_0 = m(x - x_0)$$

$$\begin{cases} P_0 \quad (\text{نقطه شروع}) \\ \vec{v} \quad \langle a,b,c\rangle \end{cases}$$

$$\vec{r} = \vec{r_0} + \vec{a}$$

$$\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + \langle ta,tb,tc\rangle$$

$$\underbrace{\langle x,y,z\rangle = \langle x_0+ta,\ y_0+tb,\ z_0+tc\rangle}$$

معادله برداری خط

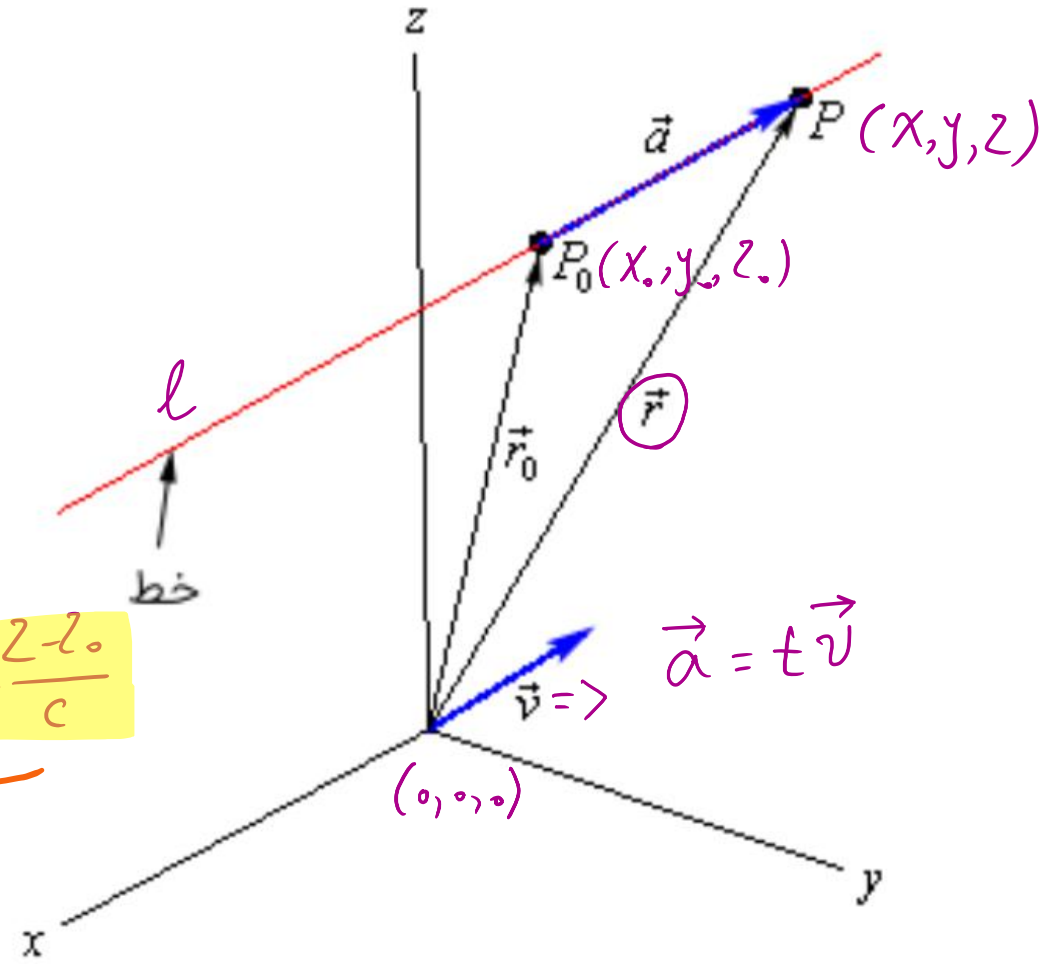
$$(x_0+ta)\vec{i} + (y_0+tb)\vec{j} + (z_0+tc)\vec{k}$$

معادله پارامتری خط

$$\begin{cases} x = x_0 + ta \Rightarrow t = \dfrac{x-x_0}{a} \\ y = y_0 + tb \Rightarrow t = \dfrac{y-y_0}{b} \\ z = z_0 + tc \Rightarrow t = \dfrac{z-z_0}{c} \end{cases}$$

$$\underbrace{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

معادلات متقارن خط



مثال) $P_0(2, 2.4, 3.5)$

$$\vec{a} = \underbrace{3}_{a}\vec{i} + \underbrace{2}_{b}\vec{j} \underbrace{-}_{c} k$$

$$r = r_0 + t v$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb + z_0 + tc \rangle$$

$$\langle 2 + 3t, 2.4 + 2t, 3.5 + (-1)t \rangle$$

معادله برداری $(2+3t)\vec{i} + (2.4 + 2t)\vec{j} + (3.5 - t)\vec{k}$

$$\vec{a} \begin{cases} A(-4, -6, 1) \\ B(-2, 0, -3) \end{cases} \quad \vec{b} \begin{cases} C(10, 18, 4) \\ D(5, 3, 14) \end{cases} \Rightarrow$$

$$\vec{a} = \langle 2, 6, -4 \rangle \quad \vec{b} = \langle -5, -15, 10 \rangle$$

$$\frac{b_1}{a_1} = \frac{-5}{2}$$

$$\frac{b_2}{a_2} = \frac{-15}{6} = \frac{-5}{2}$$

$$\frac{b_3}{a_3} = \frac{10}{-4} = \frac{-5}{2}$$

$$\vec{b} = \underbrace{\left(\frac{-5}{2}\right)}^{k} \vec{a}$$

این دو بردار موازی هستند

$l_1: \quad x = 3 + 2t \quad y = 4 - t \quad z = 1 + 3t$

$l_2: \quad x = 1 + 4s \quad y = 3 - 2s \quad z = 4 + 5s$

$\Rightarrow v_1 = \langle 2, -1, 3 \rangle$

$\Rightarrow v_2 = \langle 4, -2, 5 \rangle$

$v_1 \nparallel v_2$ موازی نیستند

فاصله نقطه P از خط را محاسبه نمایید

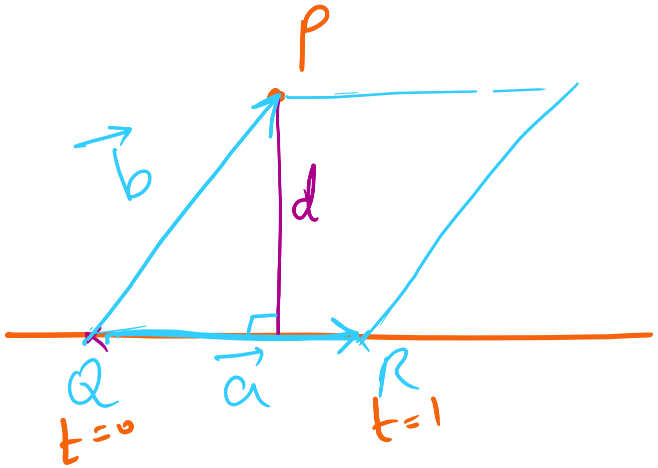
$P\,(4, 1, -2)$

$x = 1 + t$

$y = 3 - 2t$

$z = 4 - 3t$

$\rightarrow$



$|a \times b| = d \times |a| \Rightarrow d = \dfrac{|a \times b|}{|a|}$

$t = 0 \Rightarrow Q = \langle 1, 3, 4 \rangle$

$t = 1 \Rightarrow R = \langle 2, 1, 1 \rangle$

$\vec{b} = \langle 3, -2, -6 \rangle$

$\vec{a} = \langle 1, -2, -3 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix} = \underbrace{(12 - 6)}_{6}\vec{i} \overbrace{-(-6 + 9)}^{-3}\vec{j} + \underbrace{(-2 + 6)}_{4}\vec{k}$$

$$* \Rightarrow d = \frac{\sqrt{36 + 9 + 16}}{\sqrt{1 + 4 + 9}} = \sqrt{\frac{61}{14}}$$

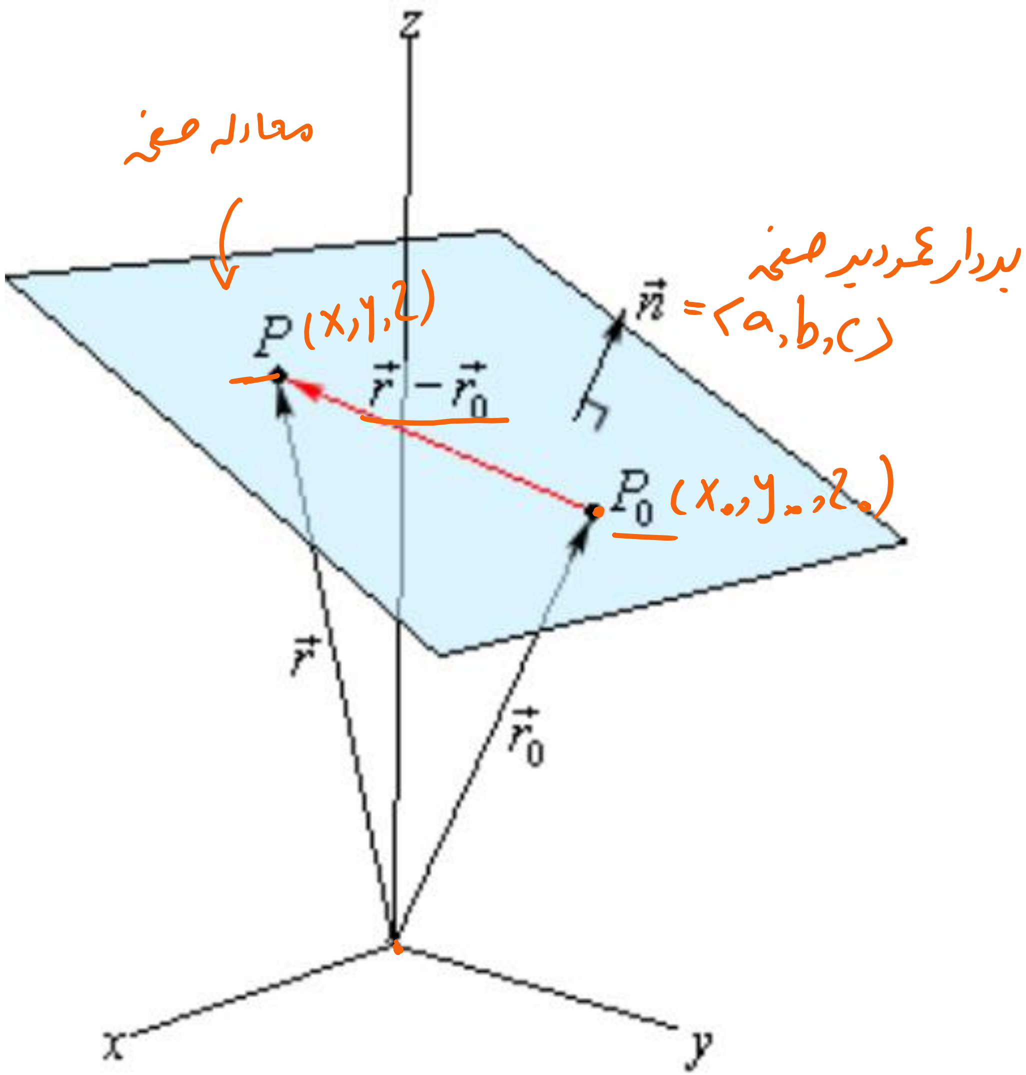
$n \perp (r - r_0) \Rightarrow$ ضرب داخلی برابر با صفر است

$$\langle a,b,c\rangle \cdot \langle x-x_0,\, y-y_0,\, z-z_0\rangle = 0$$

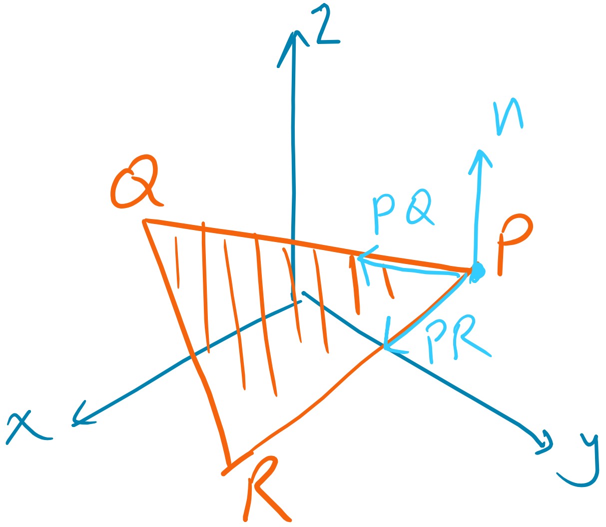
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax+by+cz\underbrace{-ax_0-by_0-cz_0}_{-(\underbrace{ax_0+by_0+cz_0}_{d})}=0$$

$$\boxed{ax+by+cz-d=0}$$



$R\,(5,2,0)$
$Q\,(3,-1,6)$
$P\,(1,3,2)$



$$\vec{PQ} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

$$n = PQ \times PR = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \underbrace{12}_{a}\vec{i} + \underbrace{20}_{b}\vec{j} + \underbrace{14}_{c}\vec{k}$$

$$ax + by + cz - d = 0$$

$$12x + 20y + 14z - (12 \times 1 + 20(3) + 14(2)) = 0$$

$$\boxed{6x + 10y + 7z - 50 = 0}$$

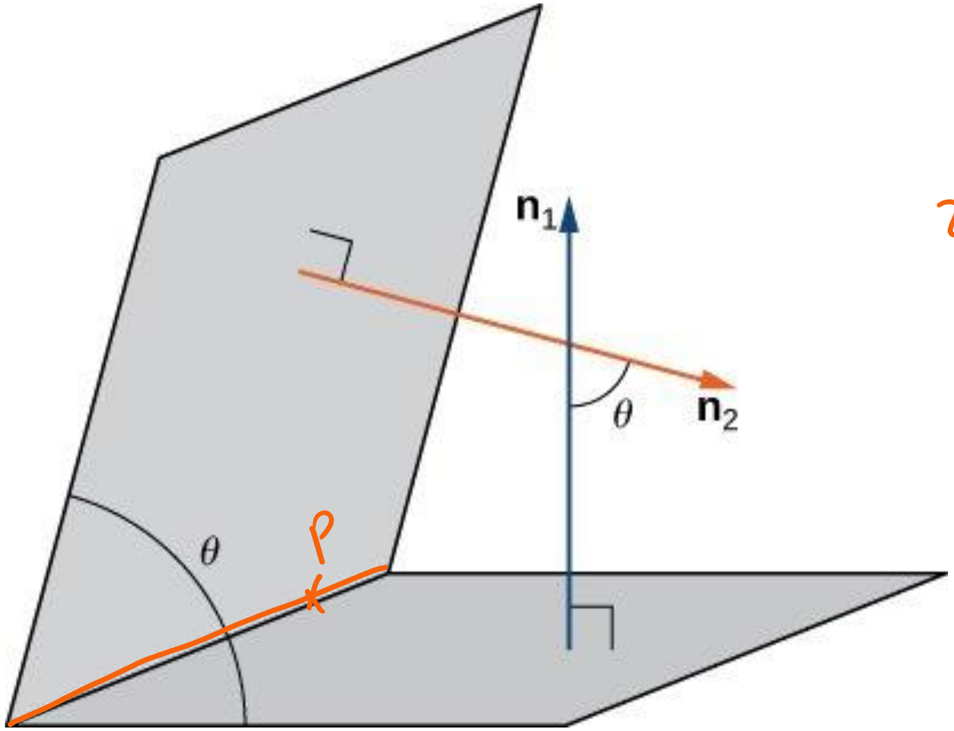
* تمرین

نقطه اشتراک خط زیر با صفحه $4x+5y-2z=18$

$$\begin{cases} x=2+3t \\ y=-4t \\ z=5+t \end{cases}$$

* معادله خط گذرنده از نقطه $(1,0,6)$ و عمود بر صفحه $x+3y+z=5$ را مشخص کنید

دوصفحه موازی و عمود و زاویه بین دوصفحه



1) $n_1 = k\, n_2$ موازی

2) $n_1 \cdot n_2 = 0$ عمود

3) $n_1 \cdot n_2 = |n_1| \cdot |n_2| \times \cos\theta$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

مثال

$$\begin{cases} x + 4y - 3z = 1 \\ -3x + 6y + 7z = 0 \end{cases}$$

$\Downarrow$

$n_1 = \langle 1, 4, -3 \rangle$ $n_2 = \langle -3, 6, 7 \rangle$ $\boxed{n \cdot n_2 = 0}$

* تمرین

زاویه بین دوصفحه

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases}$$

$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \simeq 72^\circ$

$\Downarrow$

معادله خط اشتراک

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

فاصله نقطه از صفحه

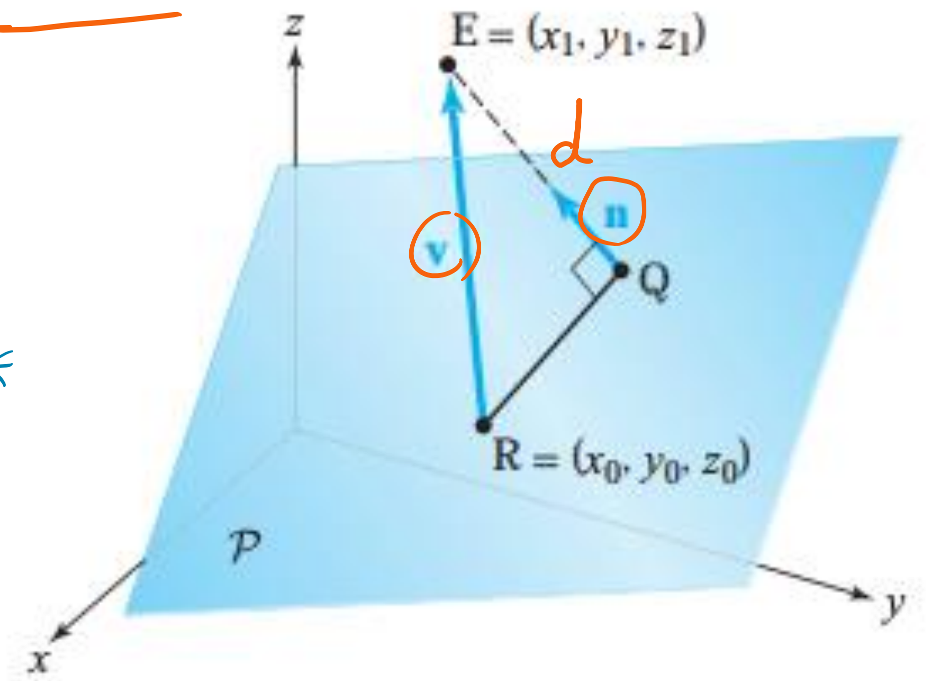
پرتو اسکالری

v در n

$$d=\frac{|\overset{\overset{(a,b,c)}{\uparrow}}{n}\cdot v|}{|n|}$$

$$v=\langle x_1-x_0,\ y_1-y_0,\ z_1-z_0\rangle$$

$$d=\frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$



* مثال: فاصله نقطه $(1,-2,4)$ از صفحه

$$\overset{a}{3}x+\overset{b}{2}y+\overset{c}{6}z=5$$

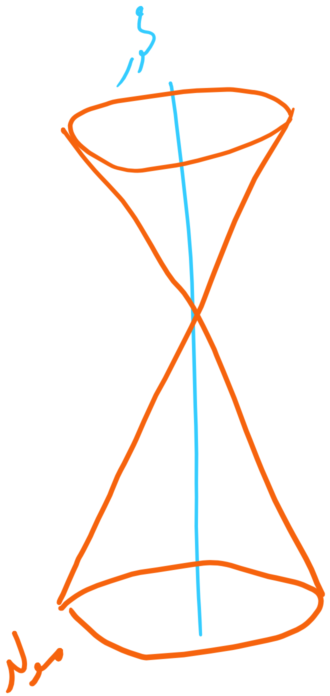
$$n=\langle 3,2,6\rangle$$

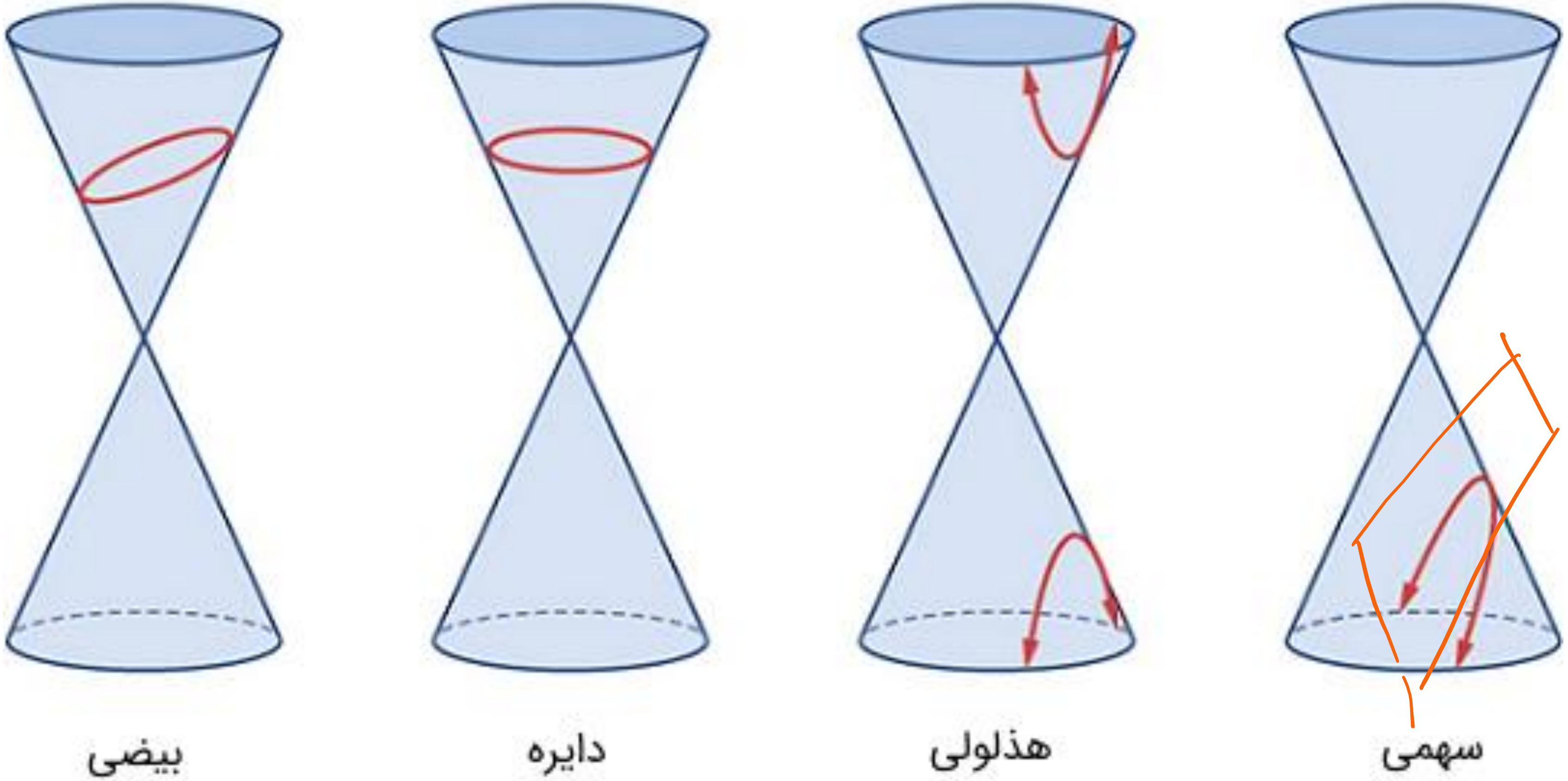
$$d=\frac{|3\times 1+2(-2)+6\times 4-5|}{\sqrt{3^2+2^2+6^2}}=\frac{18}{7}$$

$$\begin{cases} ax + by + cz + d_1 = 0 \\ ax + by + cz + d_2 = 0 \end{cases} \Rightarrow \quad d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

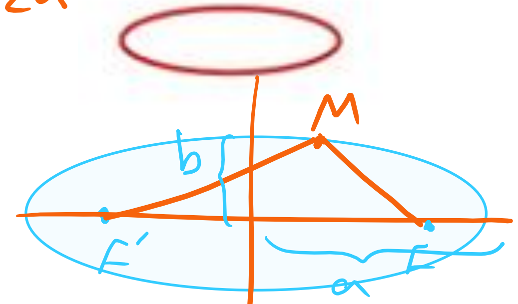
تمرین *

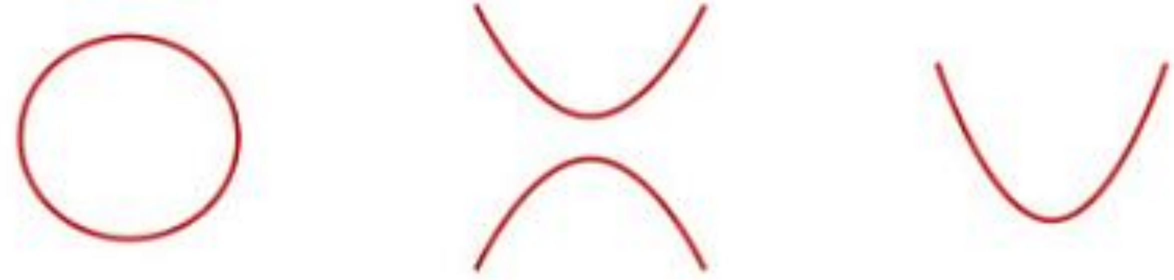
$$\begin{cases} 10x + 2y - 2z = 5 \\ 5x + y - z = 1 \end{cases} \Rightarrow$$ فاصله این دو صفحه را بیابید

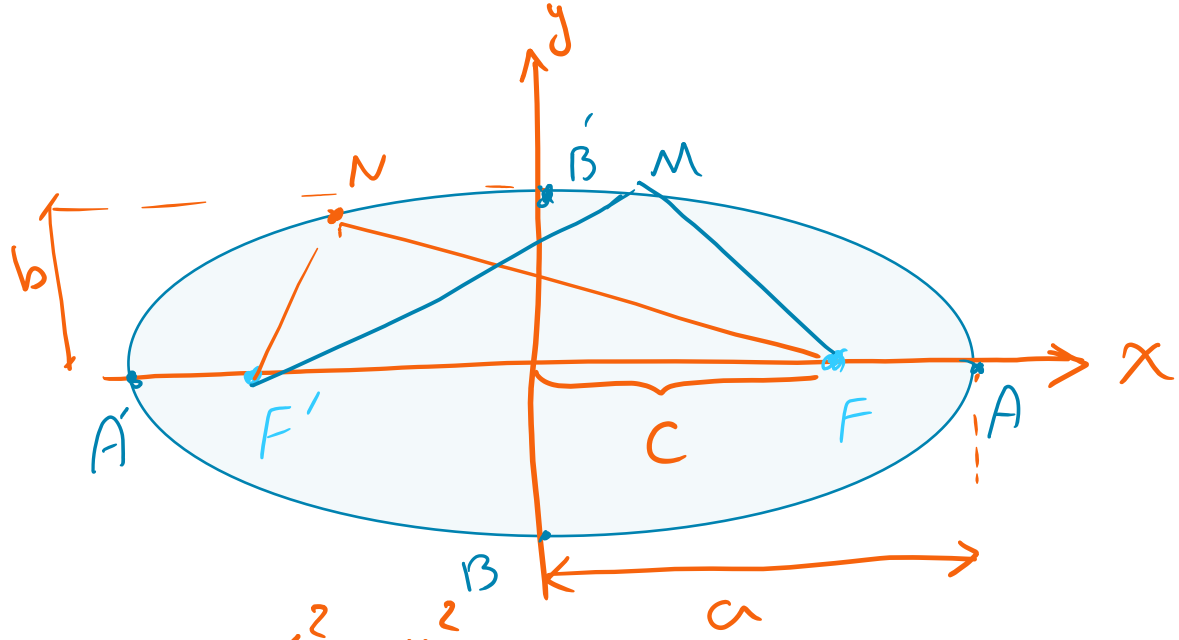




$|MF'|+|MF|=2a$



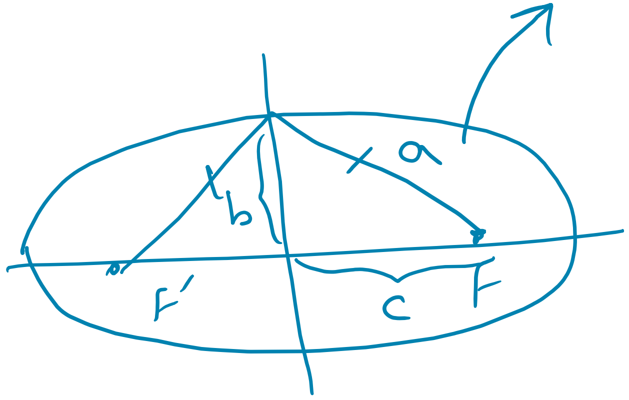




$$|MF'| + |MF| = 2a$$
$$|NF'| + |NF| = 2a$$
$$|A'A| = 2a$$
$$|B'B| = 2b$$
$$|F'F| = 2c$$

فرمول بیضی $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$a^2 = b^2 + c^2$$

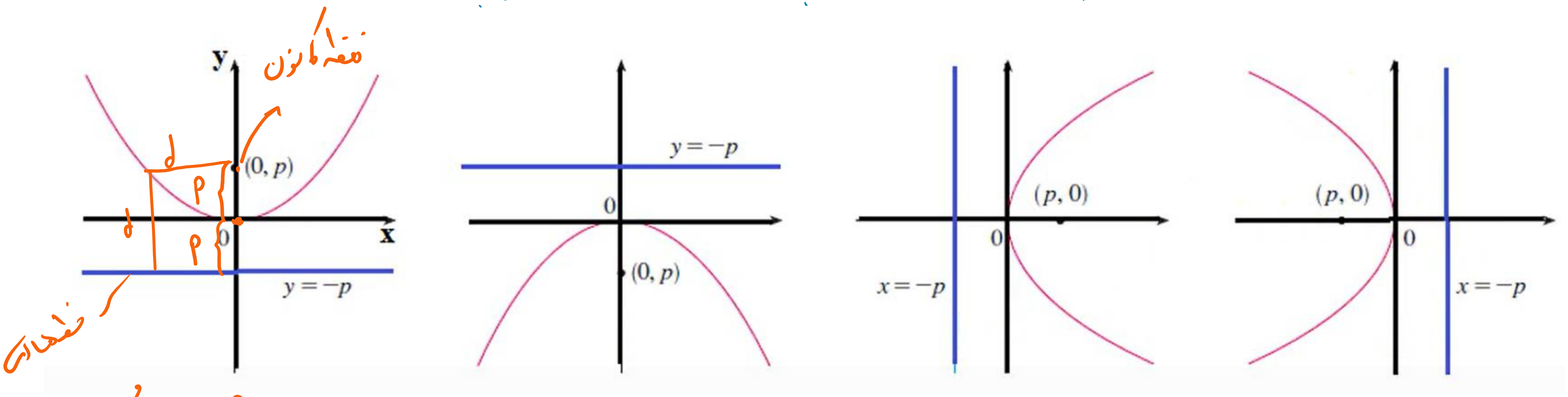


* تمرین

$9x^2 + 16y^2 = 144$ فاصله کانونی ؟

سهمی (Parabola)

مکان هندسی نقاطی که فاصله آنها از یک نقطه ثابت (کانون) و یک خط ثابت (خط هادی) مقداری برابر است



$x^2 = 4Py$ $\quad$ $x^2 = -4Py$ $\quad$ $y^2 = 4Px$ $\quad$ $y^2 = -4Px$

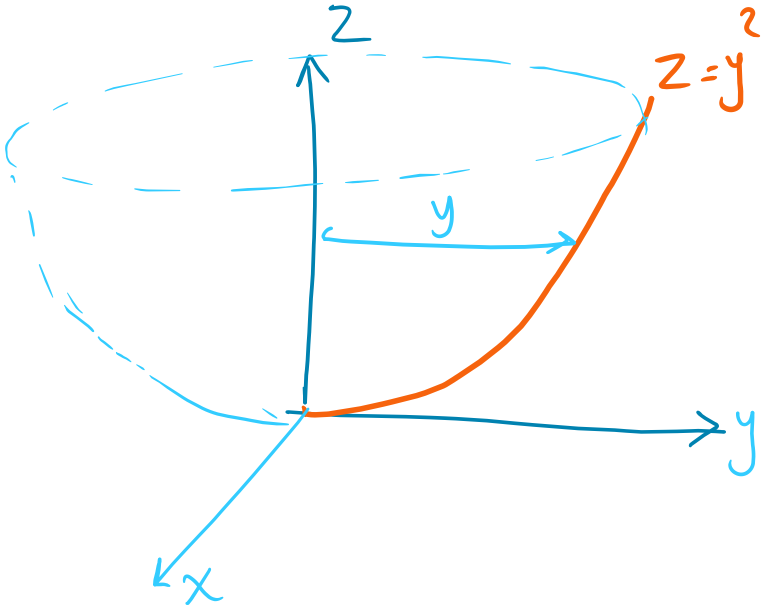
$$(y-\beta)^2 = 4P(x-\alpha)$$

مختصات مرکز (α, β)

انواع رویه (سطح)

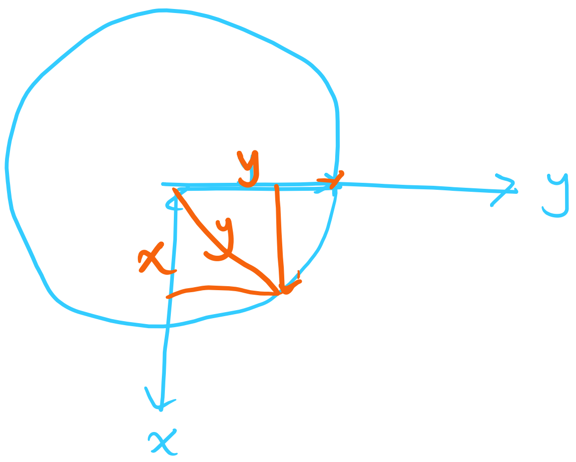
۱ـ سطوح حاصل از دوران

مثال: $\underbrace{z=y^2}_{\text{در صفحه } zy}$ این منحنی را حول محور z دوران می‌دهیم ⇐ معادله سطح ؟



شعاع دوران $\begin{cases} \sqrt{z} \\ \sqrt{x^2+y^2} \end{cases} \quad \sqrt{x^2+y^2}=\sqrt{z}$

$$\boxed{x^2+y^2=z}$$



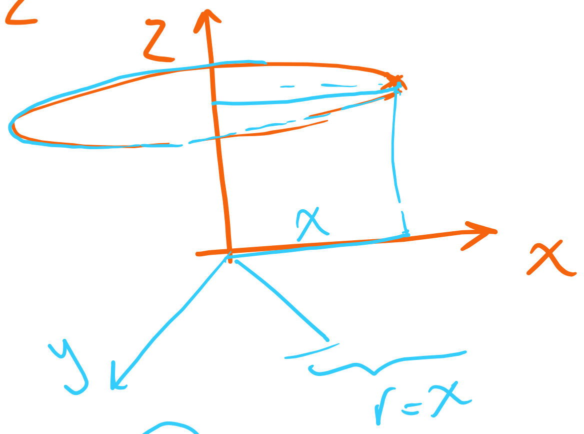
مثال:

$$\underbrace{x z^2}_{xz} = 1+\sin x^2$$

الف - دوران حول محور z {

ب - دوران حول محور x {

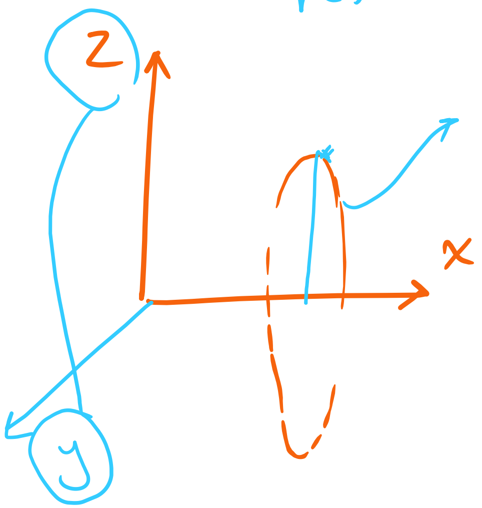
$$\left.\begin{array}{l} x \to \sqrt{x^2+y^2} \\ z \to z \end{array}\right\} \Rightarrow \left(\sqrt{x^2+y^2}\right)\times z^2 = 1+\sin\left(x^2+y^2\right)$$



$z=z$

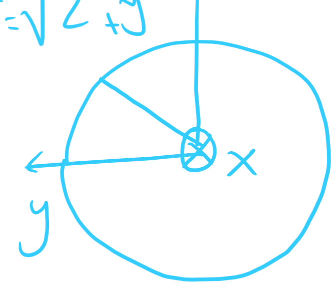
$$r=\sqrt{x^2+y^2} \xrightarrow[y=0]{zx} \boxed{r=x}$$

$r=x$

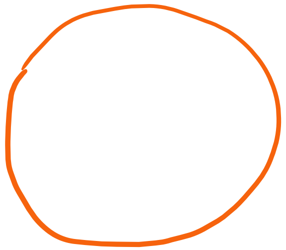


داریم $\Rightarrow \begin{cases} x=x \\ r=\sqrt{z^2+y^2} \end{cases} \xrightarrow[y=0]{zx} r=z$

$$\left.\begin{array}{l} x \to x \\ z \to \sqrt{z^2+y^2} \end{array}\right\} \Rightarrow x\left(z^2+y^2\right) = 1+\sin x^2$$



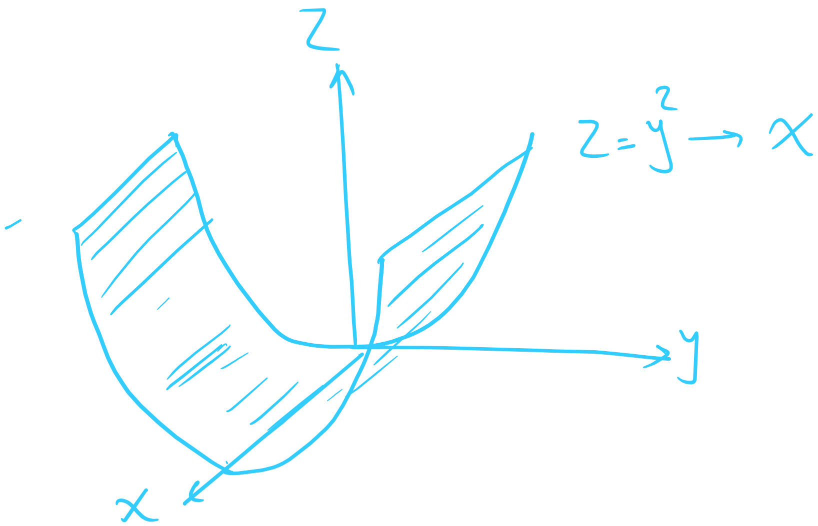
مثال: معادله رویه حاصل از دوران هذلولی $\frac{x^2}{9}-\frac{y^2}{4}=1$ حول محور x را بیابید.



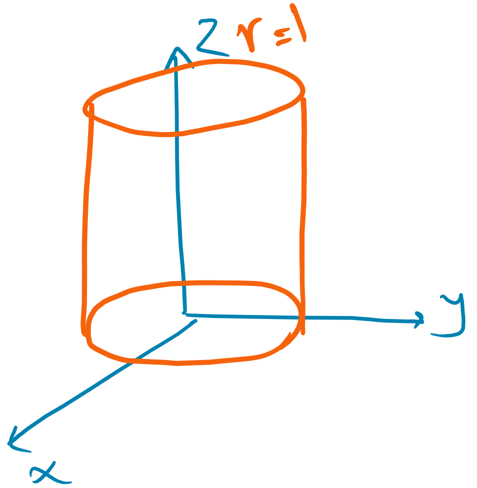
$$\left.\begin{array}{l} x \longrightarrow x \\ y \longrightarrow \sqrt{y^2+z^2} \end{array}\right\} \Rightarrow \frac{x^2}{9}-\frac{\left(\sqrt{y^2+z^2}\right)^2}{4}=1 \Rightarrow \frac{x^2}{9}-\frac{y^2}{4}-\frac{z^2}{4}=1$$

2_ استوانه

$$\mathbb{R}^3 \quad x,y,z \longrightarrow \begin{array}{ccc} x & x & y \\ y & z & x \\ \hline z & & \end{array}$$



$$\underbrace{x^2+y^2=1}_{z} \Rightarrow \begin{cases} \mathbb{R}^2 \Rightarrow \text{دایره} \\ \mathbb{R}^3 \Rightarrow \text{استوانه} \end{cases}$$



3- رویه‌های درجه 2

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

3-1) بیضی‌گون

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \xrightarrow{\times a^2b^2c^2} b^2c^2x^2+a^2c^2y^2+a^2b^2z^2=a^2b^2c^2$$

$$Ax^2+By^2+Cz^2=K$$

2-3) مخروط cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

$$x^2=4y^2+z^2$$

$$\begin{cases} x=k \Rightarrow \underline{4y^2+z^2=k^2} \Rightarrow \text{بیضی} \\ y=k \quad x^2=4k^2+z^2 \Rightarrow x^2-z^2=4k^2 \quad \text{هذلولی} \\ z=k \quad x^2-4y^2=k^2 \Rightarrow \text{هذلولی} \end{cases}$$

3-3) سهمی‌گون بیضوی

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

صفحات تقارن $\begin{cases} xoz \\ yoz \end{cases}$

مثال)

$$x^2 + 2z^2 - 6x - y + 10 = 0$$

$$x^2 - 6x + 9 + 2z^2 = y - 1$$

$$(x-3)^2 + 2z^2 = y - 1$$

تمرین)

$$x = y^2 + 4z^2 \qquad \begin{cases} x = k \\ y = k \\ z = k \end{cases}$$

$$x^2 + y^2 - 2x - 6y - z + 10 = 0$$

3-4) سهمی‌گون هذلولی

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

صفحات تقارن $\begin{cases} xoz \\ yoz \end{cases}$

3-5) هذلولی‌گون یکپارچه

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

صفحات تقارن $\begin{cases} xoz \\ yoz \end{cases}$

3-6) هذلولی‌گون دوپارچه

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

صفحات تقارن $\begin{cases} xoz \\ yoz \end{cases}$

مثال) معادله روبرو مربوط به یک مخروط است. a را بیابید

$$x^2+4y^2-6x+z^2+2z=a \qquad \underbrace{\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}} \qquad a=?$$

$$4y^2+\underbrace{x^2-6x+9}-9+z^2+2z+1-1=a$$

$$(x-3)^2-(z-1)^2+4y^2=a \Rightarrow (x-3)^2-(z-1)^2+4y^2=\underbrace{a+8}_{0} \Rightarrow a+8=0$$

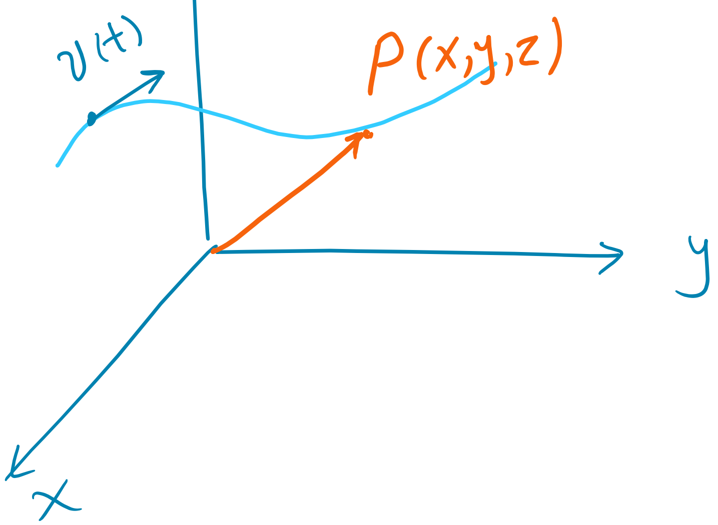
$$\boxed{a=-8}$$

خم پارامتری

رابطه در فضای دو یا چندبعدی که یک (یا چند) متغیر را گرفته و خروجی آن یک بردار است.

$$\vec{\alpha}(t) = (x(t), y(t), z(t)) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \qquad \mathbb{R} \to \mathbb{R}^3$$

بردار مکان (موضع) $\vec{\alpha} = \vec{OP}$



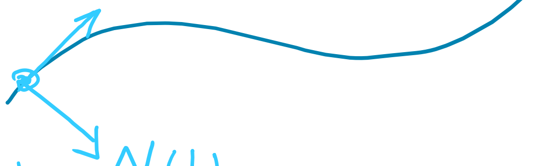
$$\vec{v}(t) = \alpha'(t) = (x'(t), y'(t), z'(t))$$ بردار سرعت (مماس بر مسیر حرکت)

$$|\vec{v}(t)| = \sqrt{x'^2 + y'^2 + z'^2}$$ تندی

$$\vec{a}(t) = v'(t) = (x''(t), y''(t), z''(t))$$

بردار یکه مماس $T(t)$

$\alpha(t)$



بردار قائم واحد اصلی $N(t)$

$$\underbrace{\vec{B}(t)} = \vec{T}(t) \times \vec{N}(t)$$

بردار قائم واحد دوم

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$\vec{N}(t) = \frac{T'(t)}{|T'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$