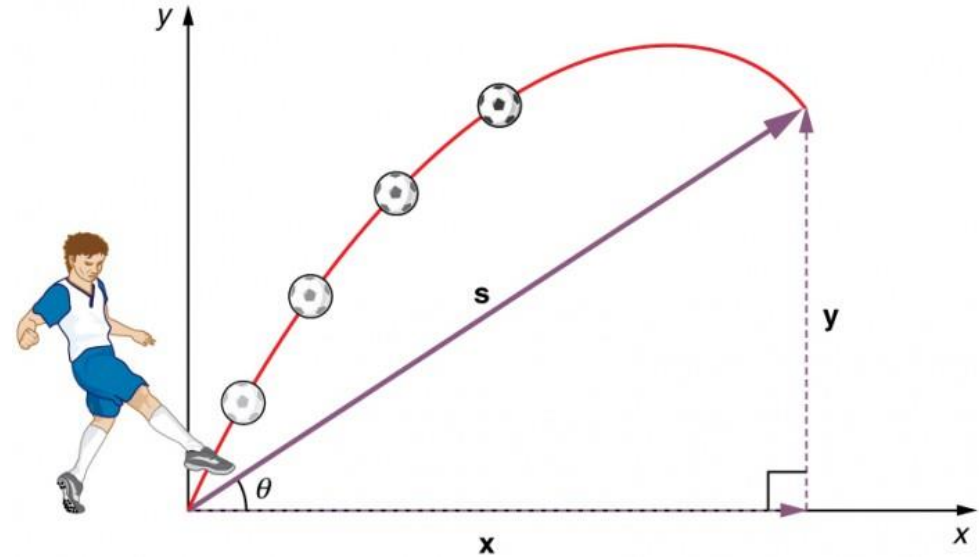
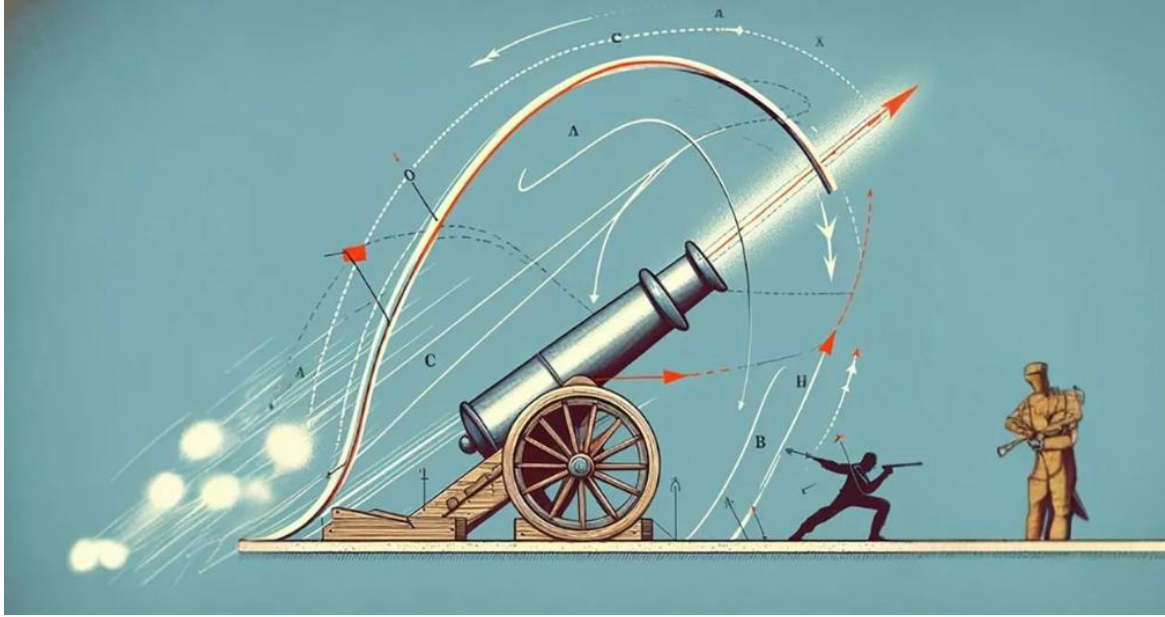




$$x(t) = V_0 t \cos(\theta)$$

$$y(t) = -\frac{1}{2}gt^2 + V_0 t \sin\theta$$

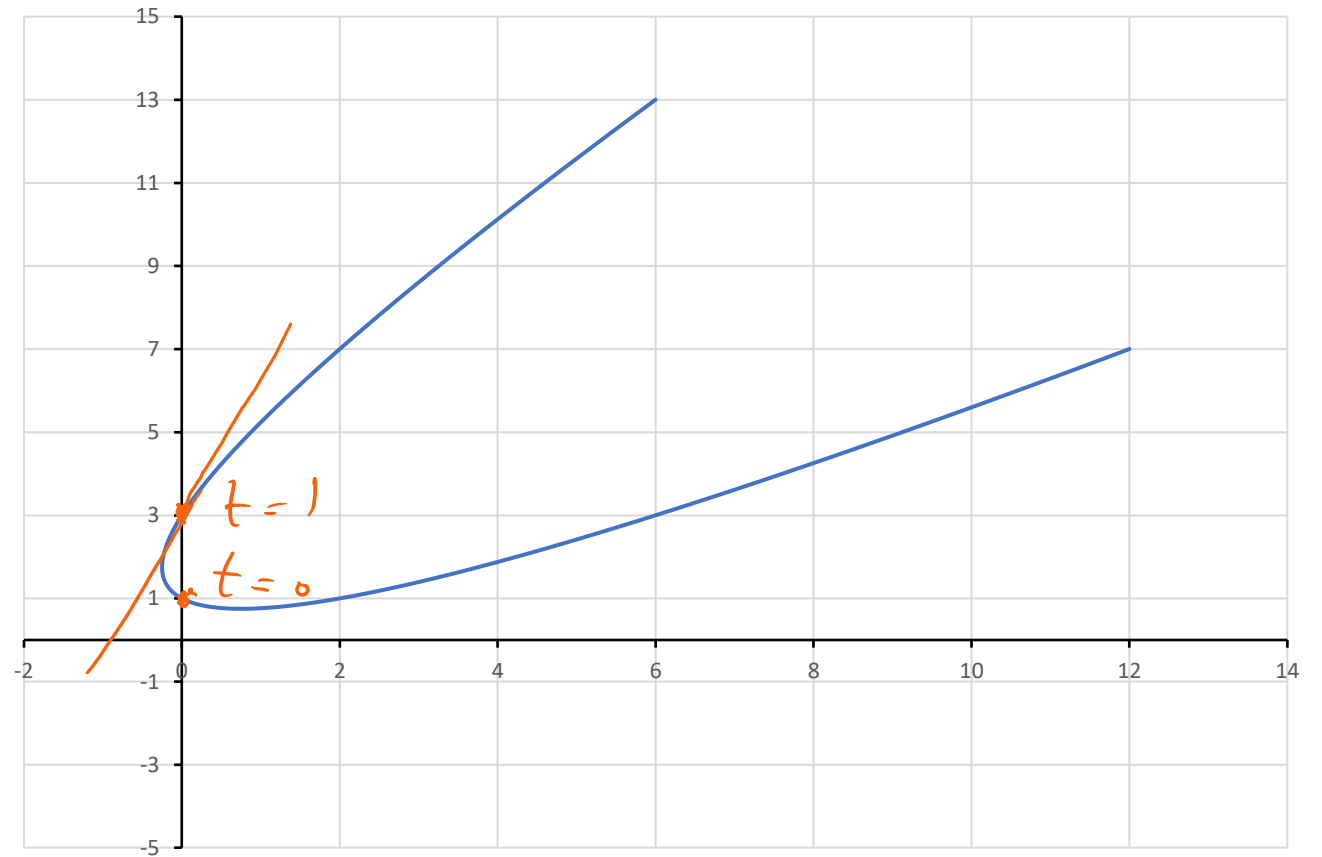
$$x = t^2 - t \quad \rightarrow \quad 0 = t^2 - t \Rightarrow t(t-1) = 0 \quad \left\{ \begin{array}{l} t=0 \\ t=1 \end{array} \right.$$

$$y = t^2 + t + 1$$

$$(x, y) = (0, 3)$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t-1} = \frac{3}{1} = 3$$

$$\underline{y = 3x + 3}$$



$$x = t^2 - 2t = 8$$

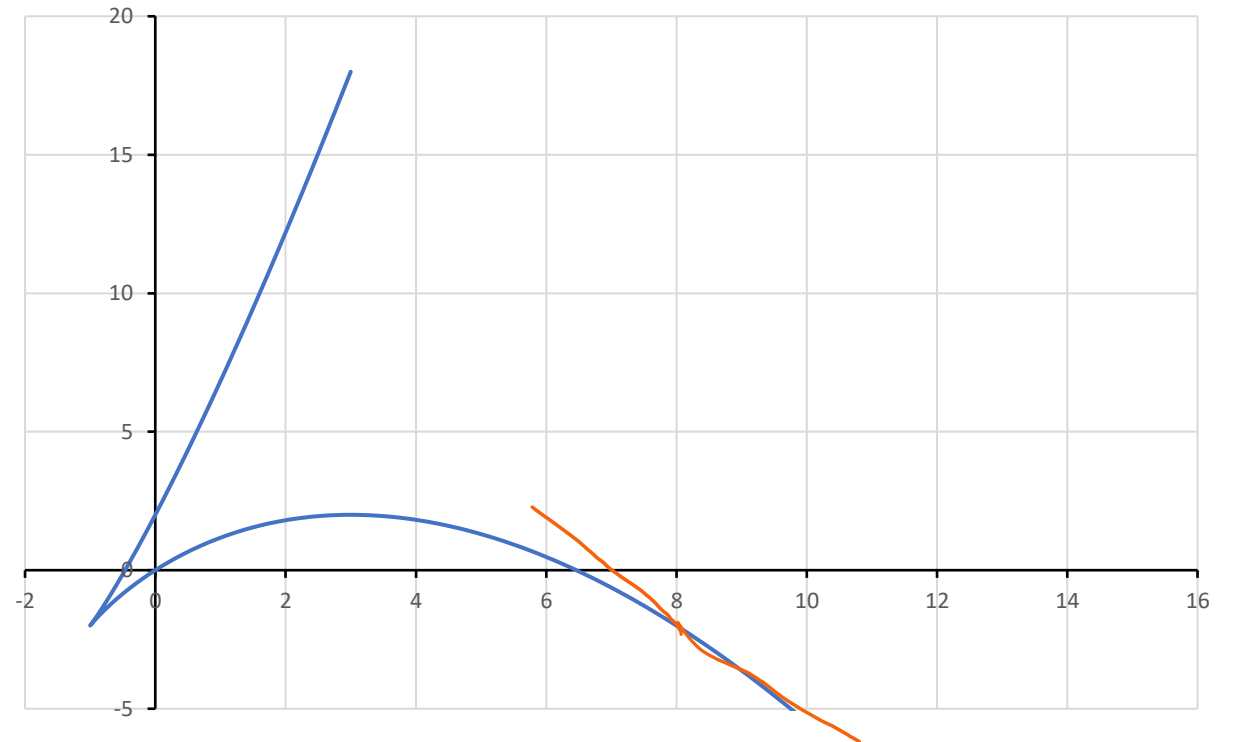
$$y - (-2) = m(x - 8)$$

$$y = t^3 - 3t \quad -8 + 6 = -2$$

$$t = -2$$

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t - 2} = \frac{12 - 3}{-6} = \frac{+9}{-6} = -\frac{3}{2}$$

$$y = -\frac{3}{2}(x - 8) - 2 = -\frac{3}{2}x + 10$$



$$x = t^2$$

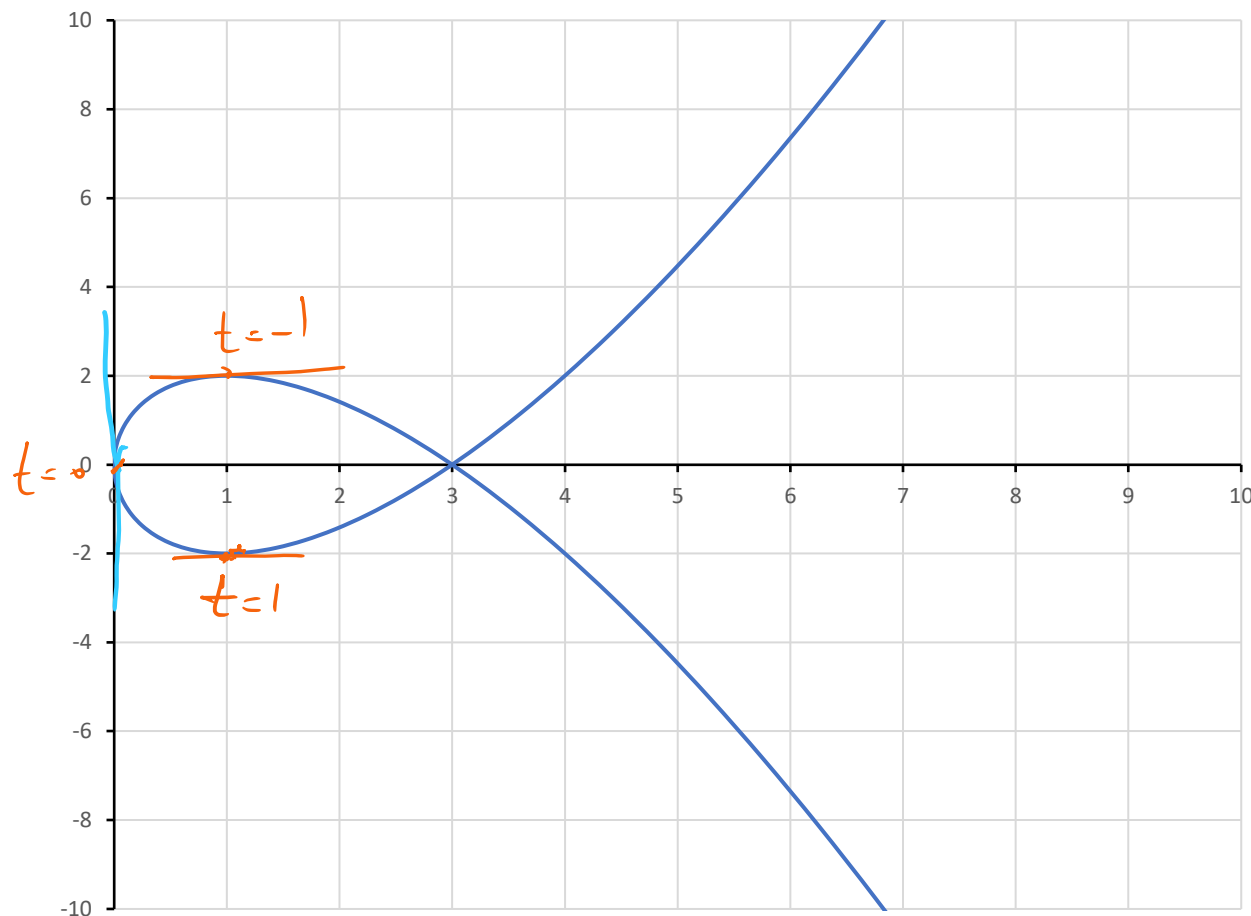
$$y = t^3 - 3t$$

نقاط مفردة

$$\frac{dx}{dt} = 0, \frac{dy}{dt} \neq 0$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dx}{dt} \neq 0, \frac{dy}{dx} = 0$$



$$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2 - 1) \Rightarrow \begin{cases} t = -1 \\ t = 1 \end{cases}$$

$$\frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \Rightarrow y = 0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow \begin{cases} t = 0 \\ t = \pm\sqrt{3} \end{cases}$$

$$m = \frac{3t^2 - 3}{2t} \Rightarrow \frac{6}{2\sqrt{3}} = \sqrt{3}, -\sqrt{3}$$

$$t = \sqrt{3} \Rightarrow x = 3, y = (\sqrt{3}^3 - 3\sqrt{3})$$

$$x = r(\theta - \sin(\theta))$$

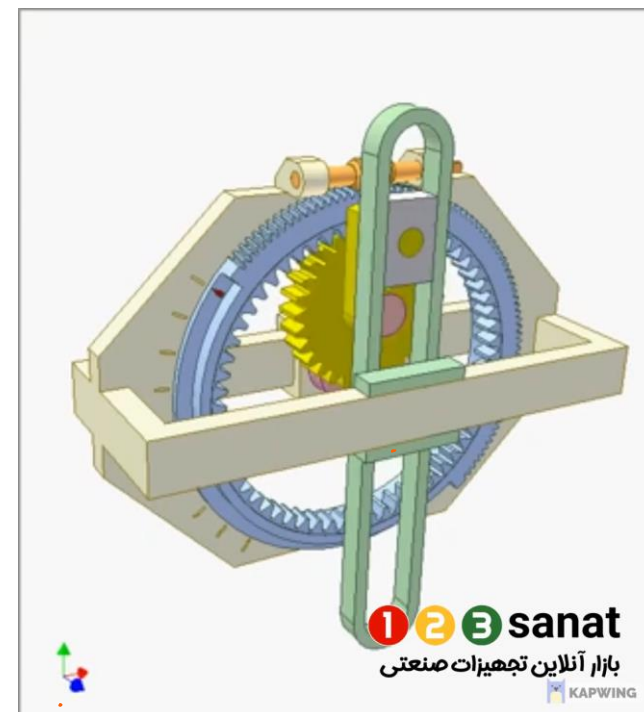
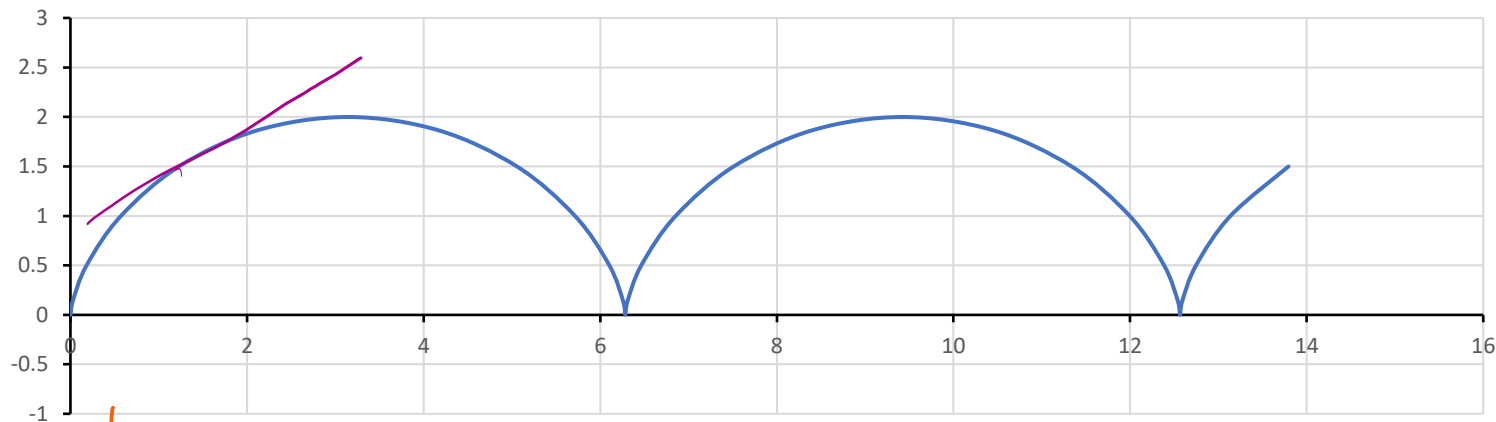
$$y = r(1 - \cos(\theta))$$

$-\sin \theta$

$$\theta = \frac{\pi}{3}$$

$$m = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\sin \theta)}{r(1 - \cos \theta)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \quad x, y$$

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \quad y = r\left(\frac{1}{2}\right) \frac{r}{2} \quad y - \frac{r}{2} = \sqrt{3} \left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$



$$x = t^2 + 1$$

$$y = t^2 + t$$

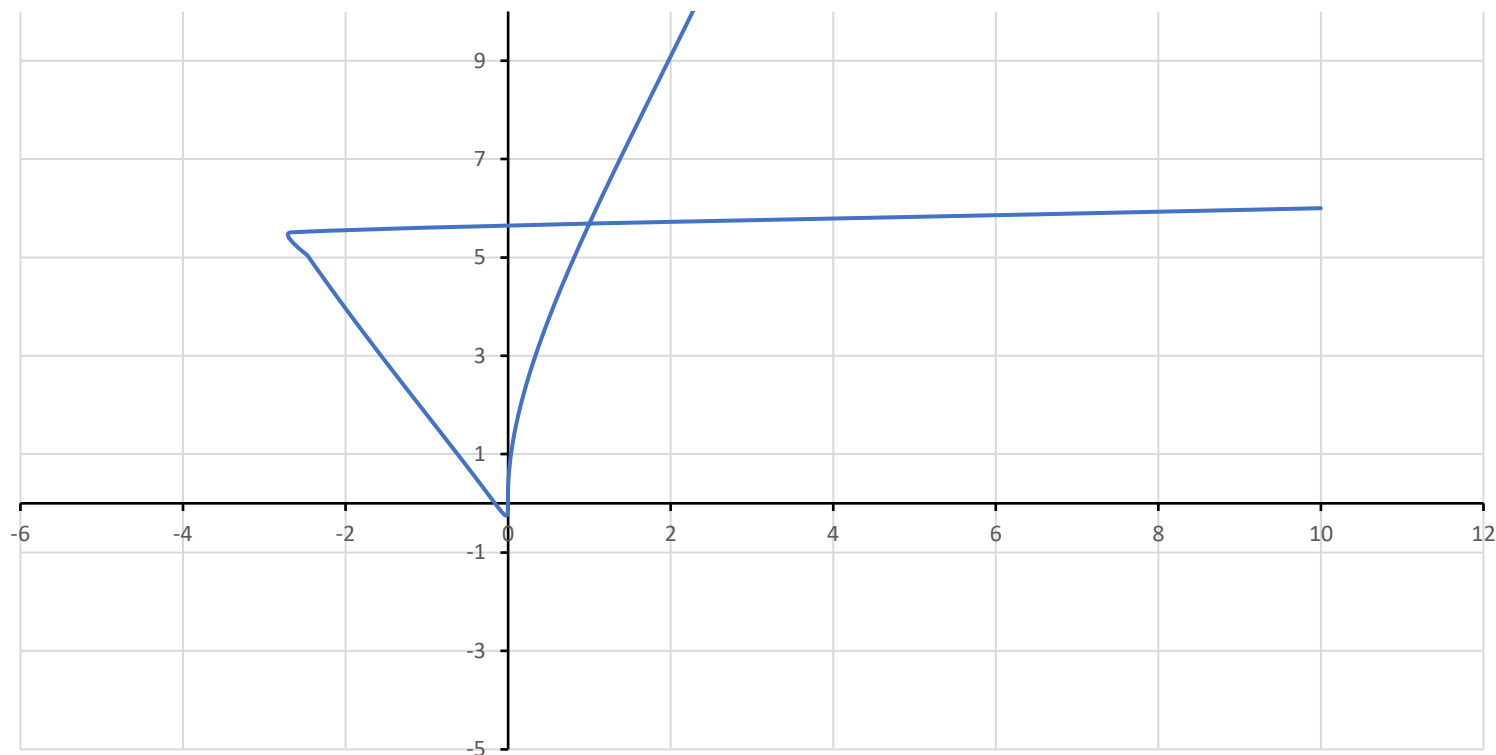
$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t}$$

$$\frac{d}{dt} \left(\frac{\frac{dy}{dx}}{\frac{dx}{dt}} \right)$$

$$\frac{4t - 4t}{2(2t)^2 - 2(2t+1)}$$

$$\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{-2}{8t^3} = \frac{-1}{4t^3} \Rightarrow \text{منفرد}$$



$$x = t^2$$

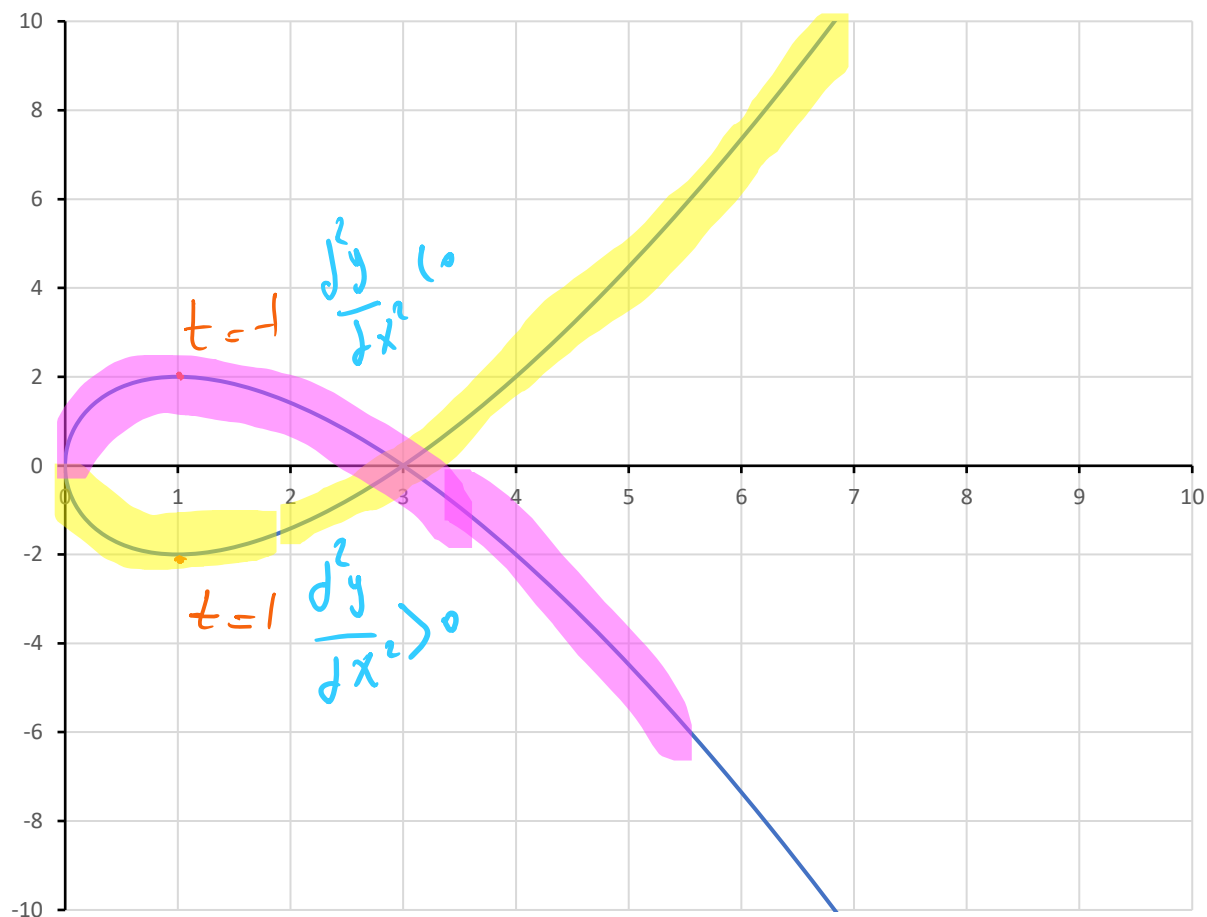
$$y = t^3 - 3t$$

$$\frac{d^2 y}{dx^2} = ? \quad \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dt} = 3t^2 - 3 \quad \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} \xrightarrow{\frac{d}{dt}} \frac{12t^2 - 6t^2 + 6}{4t^2} = \frac{6t^2 + 6}{4t^2}$$

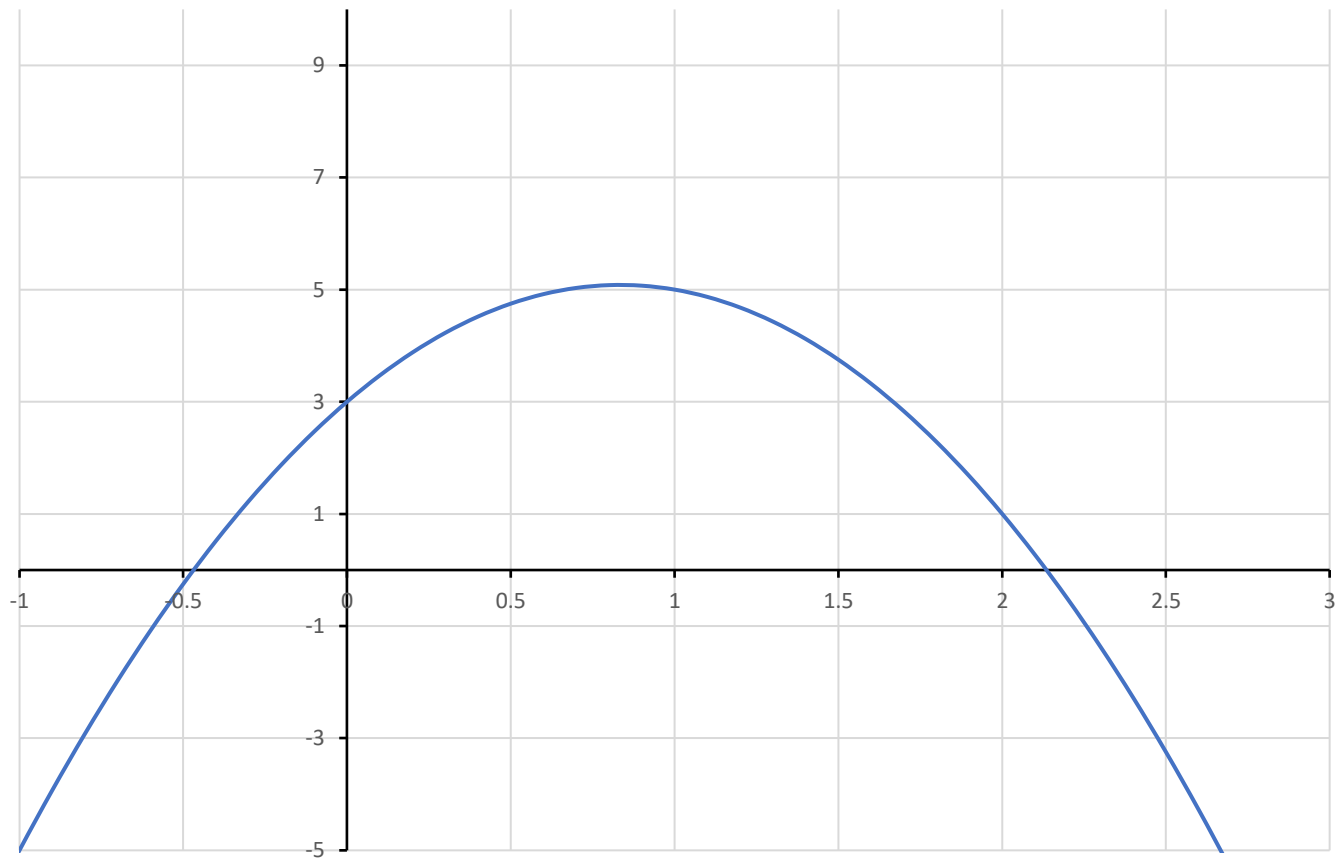
$$\frac{d^2 y}{dx^2} = \frac{\frac{6t^2 + 6}{4t^2}}{2t} = \frac{6(t^2 + 1)}{8t^3} = \frac{3(t^2 + 1)}{4t^3}$$



$$A = \int_a^b y \, dx = \int_\alpha^\beta g(t)f'(t)dt$$

$$x = f(t) \Rightarrow dx = f'(t)dt$$

$$y = g(t)$$



$$A = \int_a^b y \, dx = \int_\alpha^\beta \overbrace{g(t)}^{f(t)} f'(t) \, dt$$

$$x = t^3 + 1 \Rightarrow dx = 3t^2 dt \quad 0 < t < 2$$

$$y = 2t - t^2$$

$$A = \int_0^2 (2t - t^2) 3t^2 \, dt = 3 \int_0^2 (2t^3 - t^4) \, dt$$

$$= 3 \left[\frac{2t^4}{4} - \frac{t^5}{5} \right]_0^2 = \frac{24}{5}$$



$$x = r(\theta - \sin(\theta))$$

$$y = r(1 - \cos(\theta))$$

$$dx = r(1 - \cos \theta) d\theta$$

$$0 < \theta < 2\pi$$

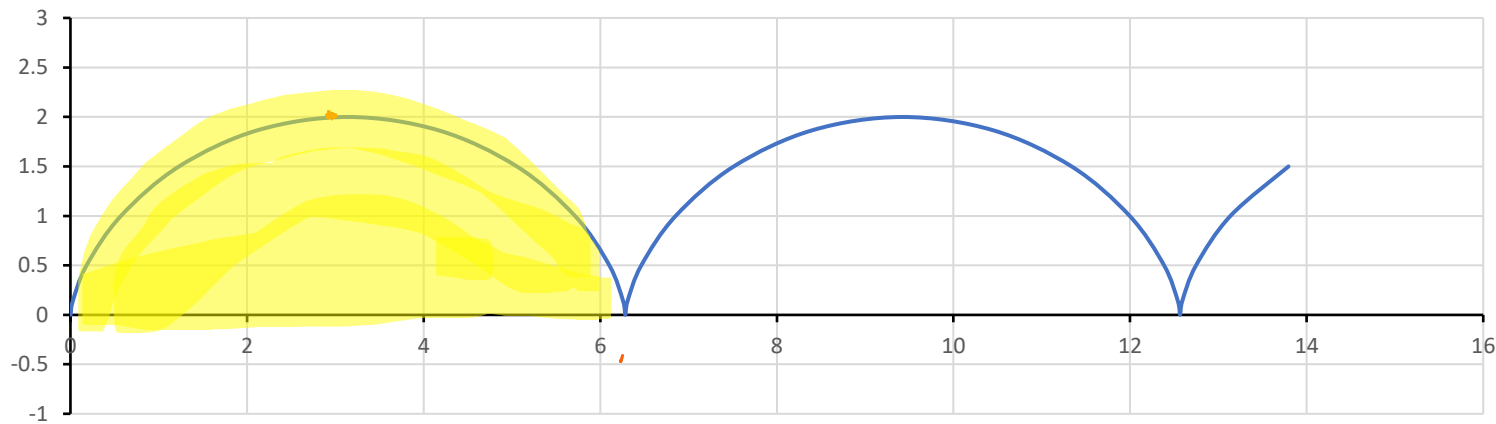
$$A = \int y dx = \int_0^{2\pi} g(\theta) f'(\theta) d\theta$$

$$A = \int r^2 (1 - \cos \theta)(1 - \cos \theta) d\theta$$

$$A = r^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = r^2 \int_0^{2\pi} (1 + \cos^2 \theta - 2\cos \theta) d\theta = r^2 \left[\frac{3}{2} \theta + \frac{1}{2} \left(\frac{1}{2} \sin 2\theta \right) - 2\sin \theta \right]_0^{2\pi}$$

$$r^2 \int_0^{2\pi} \left[\frac{3}{2} + \frac{1}{2} \cos 2\theta - 2\cos \theta \right] d\theta$$

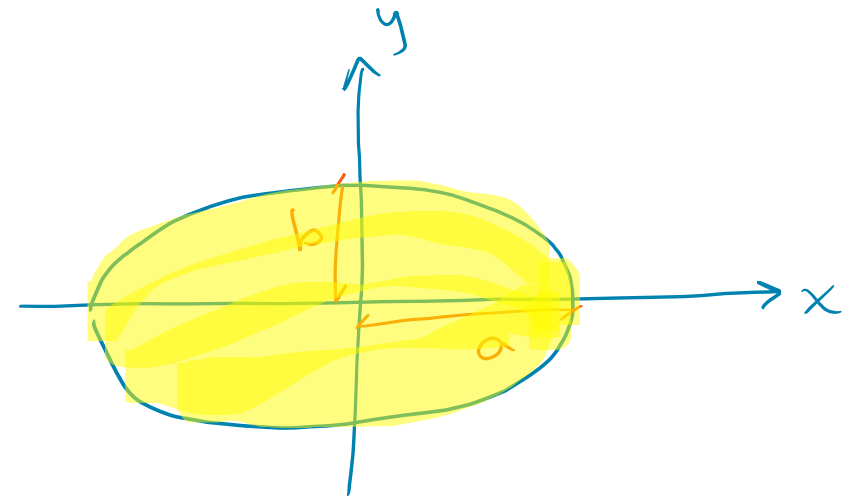
$$= r^2 \left[\frac{3}{2} (2\pi) \right] = \underline{3\pi r^2}$$



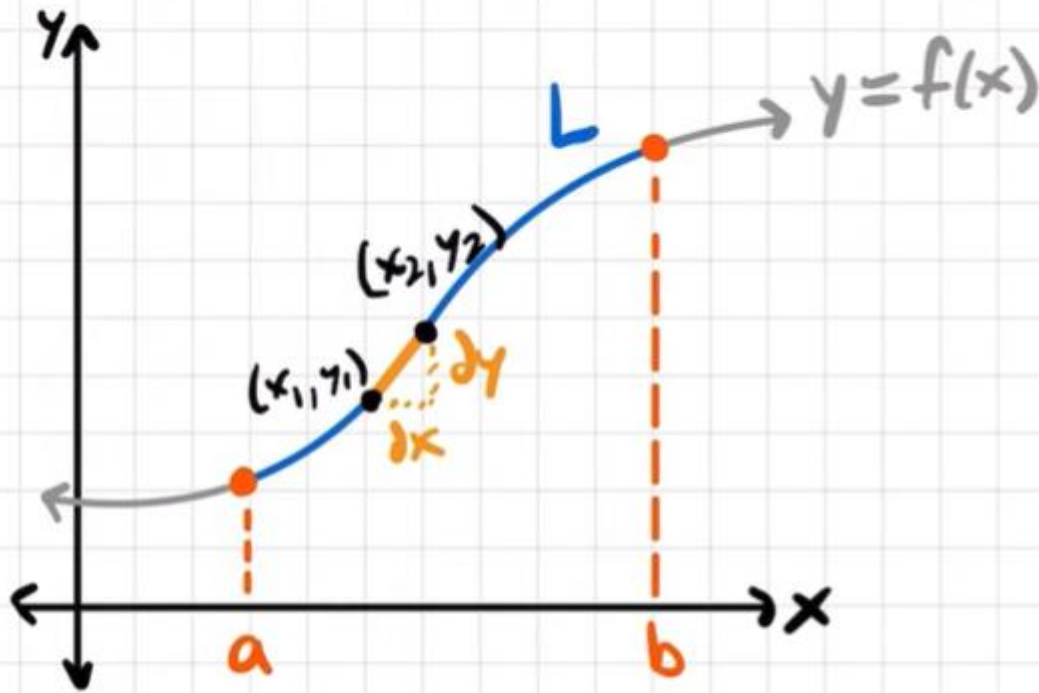
$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned}$$

$$\left. \begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \right\} \begin{aligned} \frac{x}{a} &= \cos \theta \\ \frac{y}{b} &= \sin \theta \end{aligned}$$

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$



Lesson 6: Arc Length



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$
$$= \int_a^b \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

$$= \int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

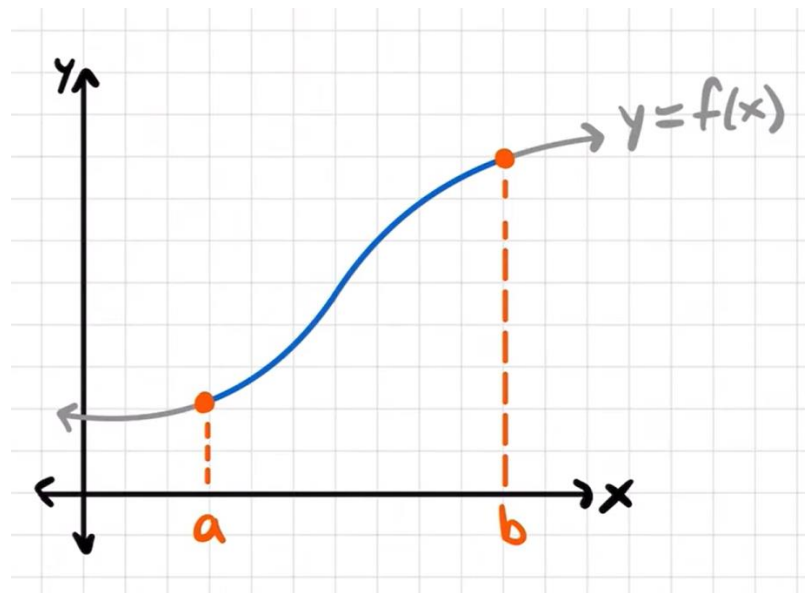
$$= \int_a^b \sqrt{1 + (f'(x))^2} dx$$



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

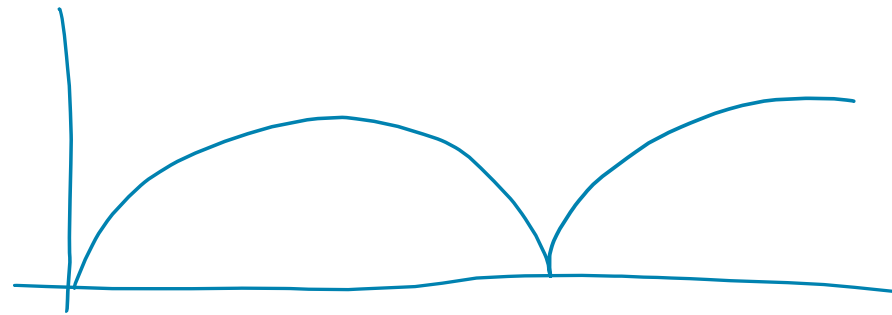
$$x = f(t)$$

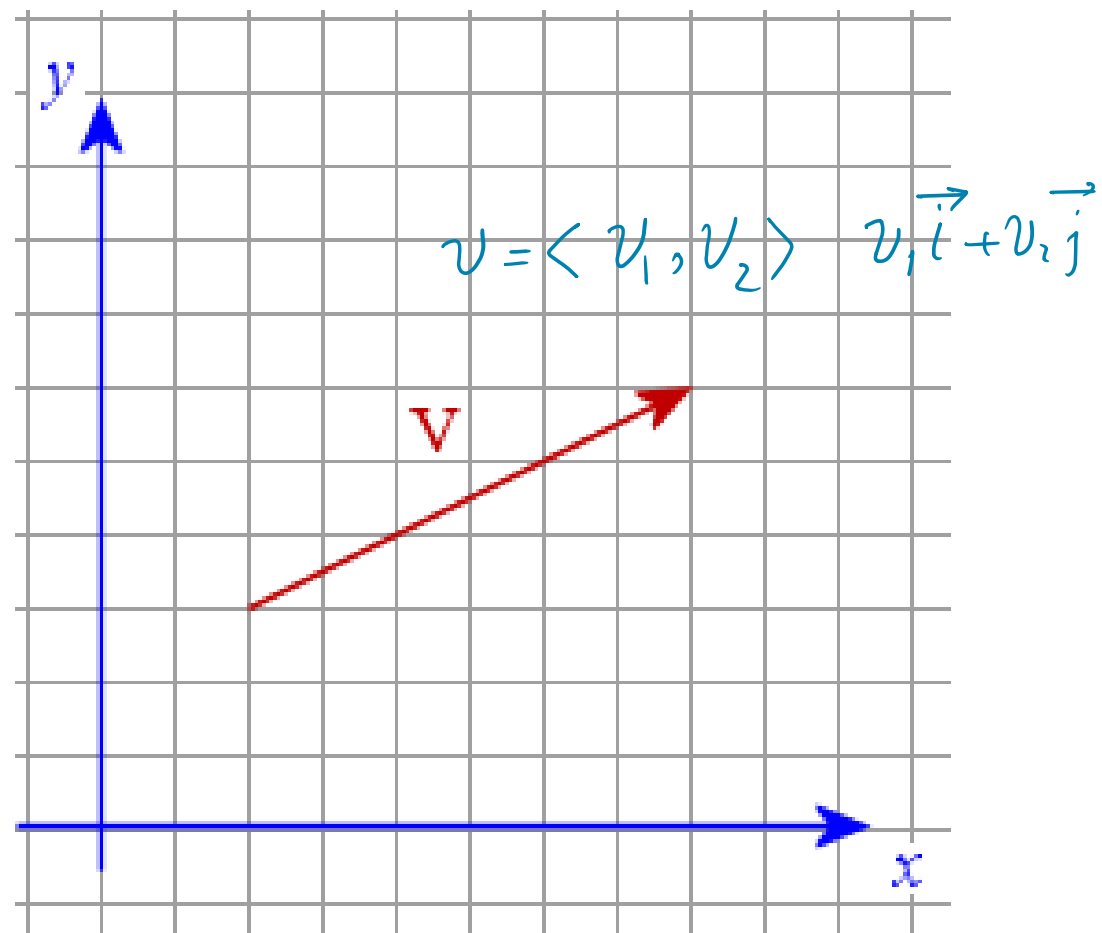
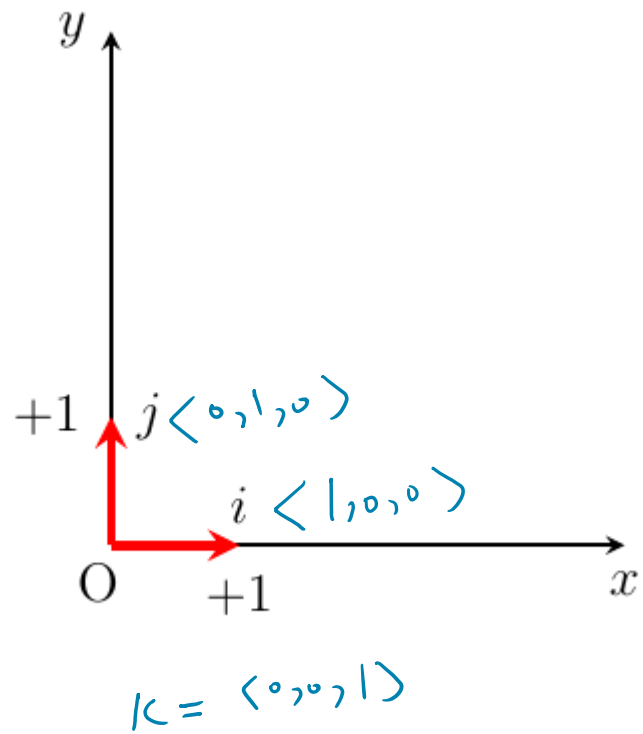
$$y = g(t)$$

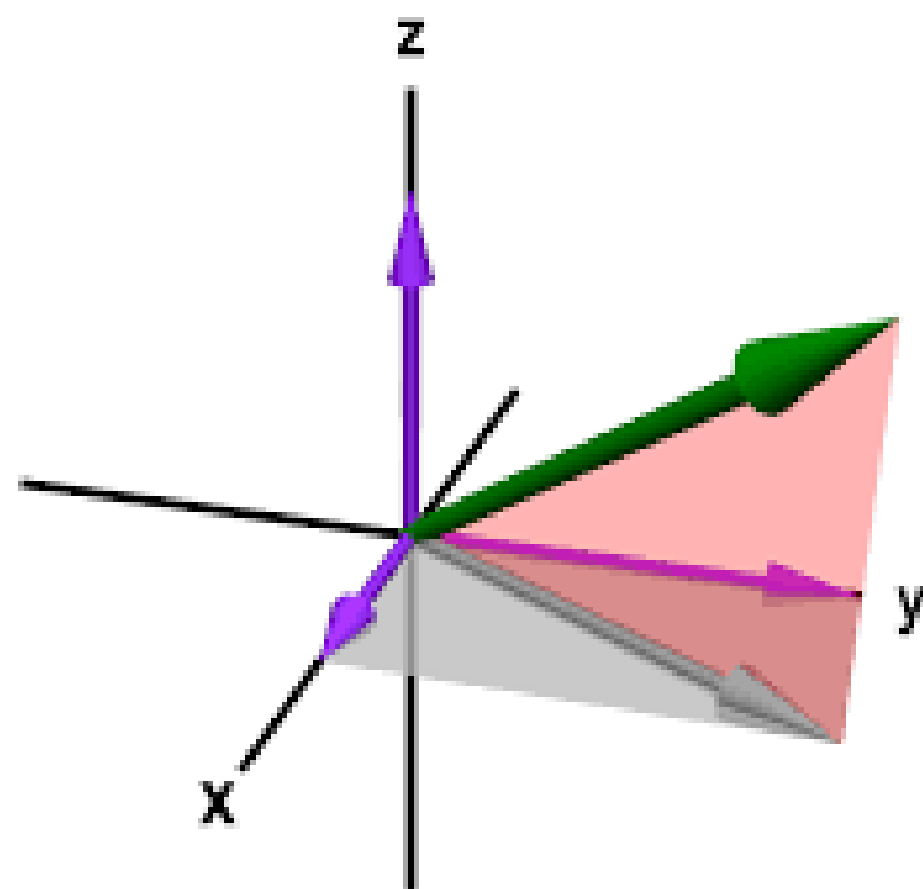
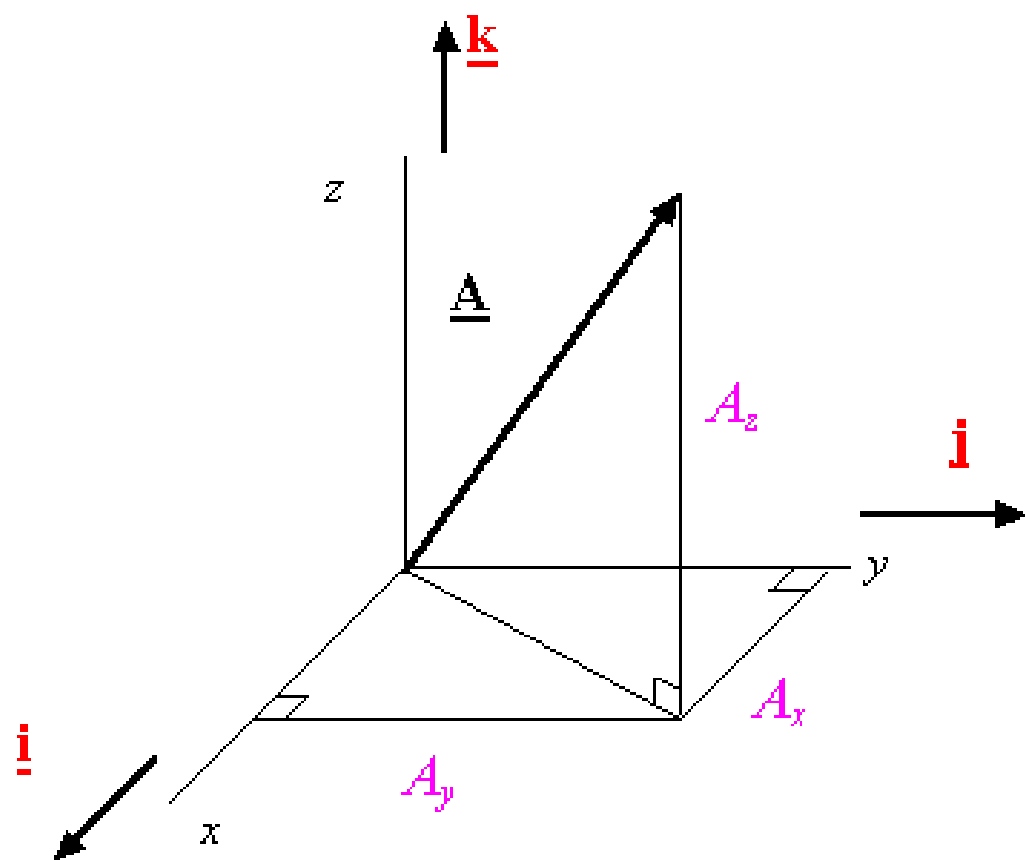


$$L = \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \times \frac{dx}{dt} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{cases} x = r(1 - \sin \theta) \Rightarrow \frac{dx}{d\theta} \\ y = r(1 - \cos \theta) \Rightarrow \frac{dy}{d\theta} \end{cases} \Rightarrow \text{ok!}$$







مثال: بردار واحد هم جهت با بردار $\langle -2, 6 \rangle$ را پیدا کنید.

$$\vec{v} = 6\vec{i} - 2\vec{j}$$

$$\left\langle \frac{v_1}{|\vec{v}|}, \frac{v_2}{|\vec{v}|} \right\rangle =$$

$$|\vec{v}| = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$$

$$\left\langle \frac{6}{2\sqrt{10}}, \frac{-2}{2\sqrt{10}} \right\rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\left\{ \frac{\vec{v}}{|\vec{v}|} \Rightarrow \text{بردار واحد} \right.$$

$$a = \langle a_1, a_2 \rangle$$

$$b = \langle b_1, b_2 \rangle$$

$$a+b = \langle a_1+b_1, a_2+b_2 \rangle$$

$$a+(b+c) = (a+b)+c$$

$$a+0 = a$$

$$a+(-a) = 0$$

$$c(a+b) = ca+cb$$

$$(cd)a = c(da)$$

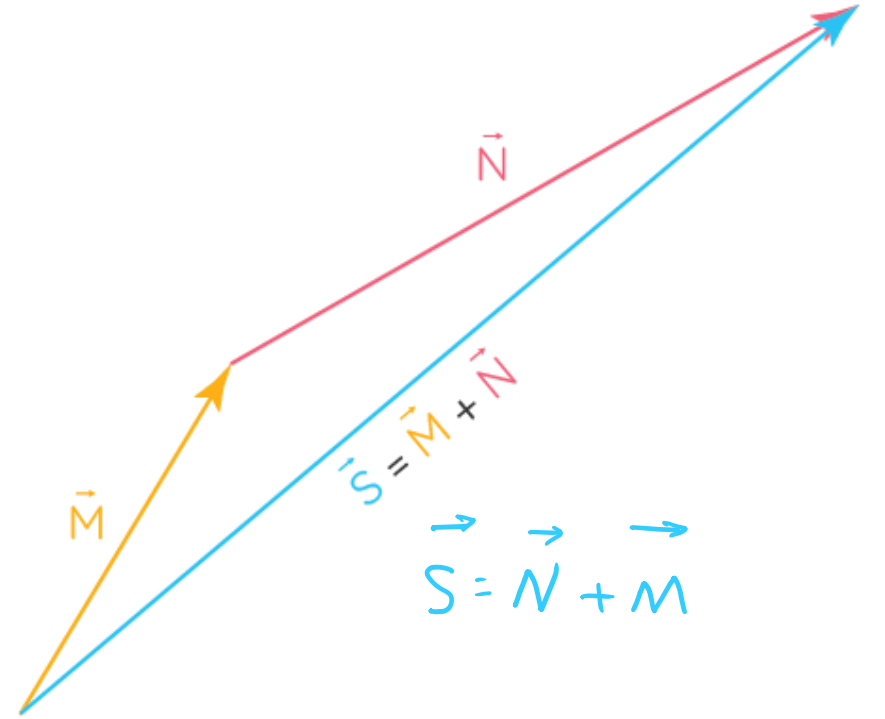
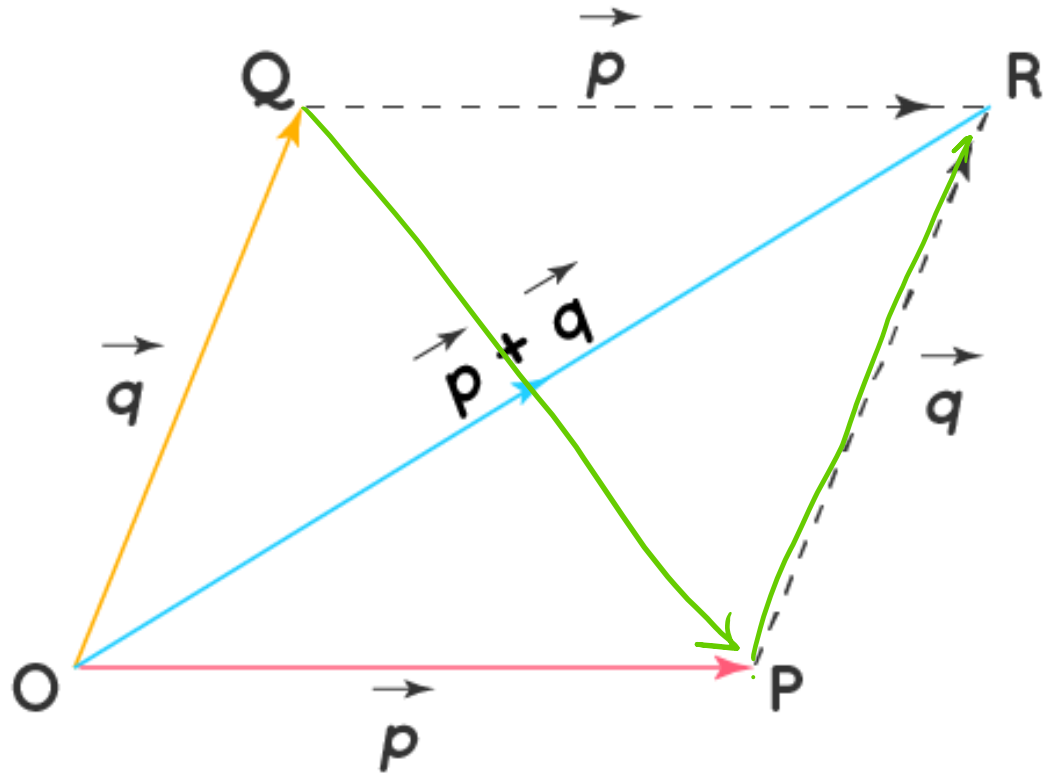
$$1a = a$$

$$a+b = b+a \quad \text{خاصیت جابجایی دارد}$$

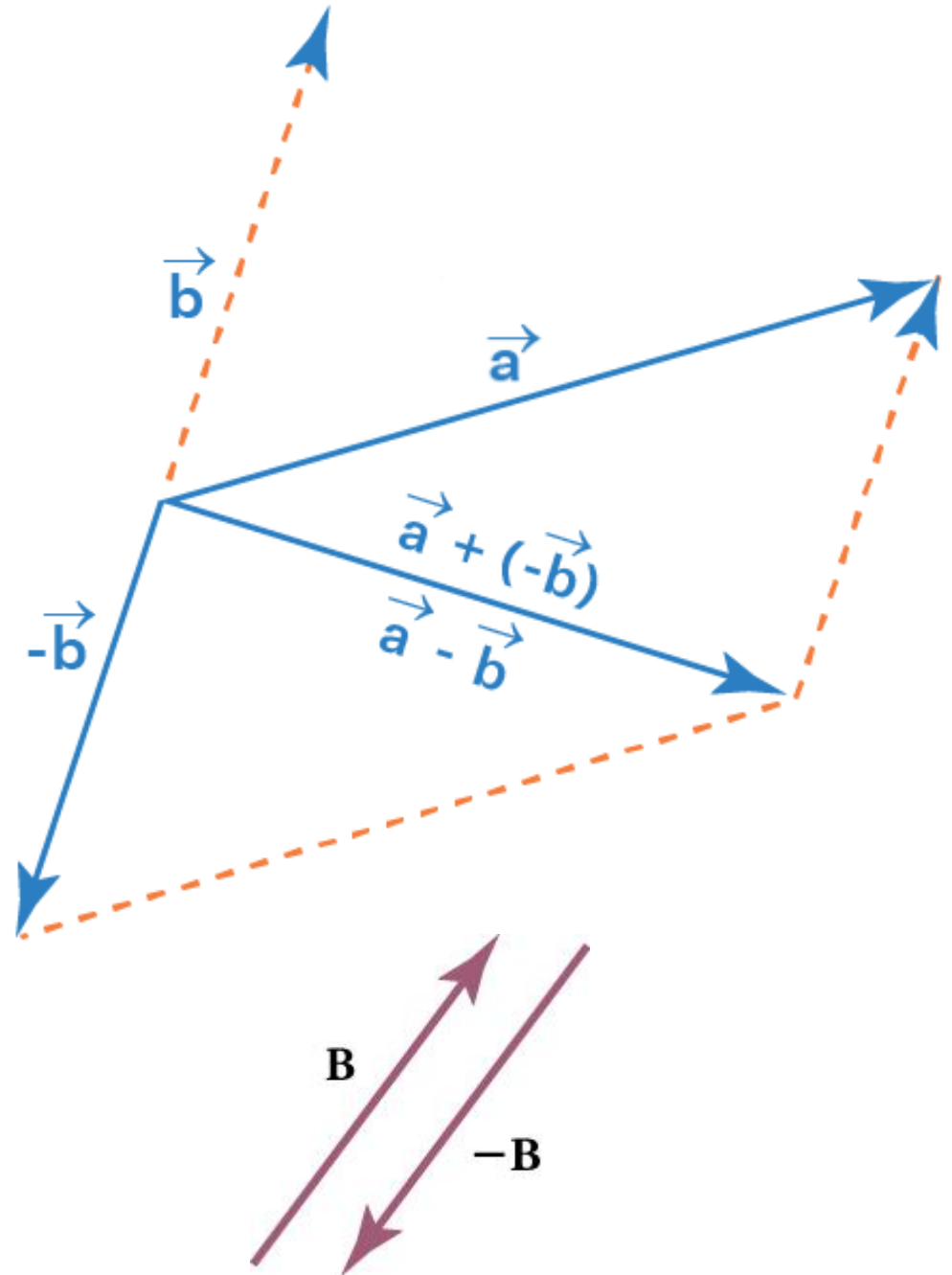
جهت
بردار

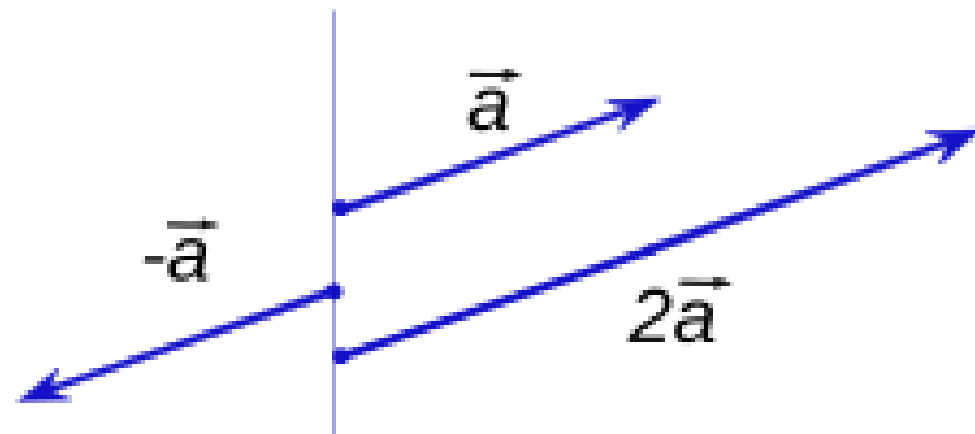
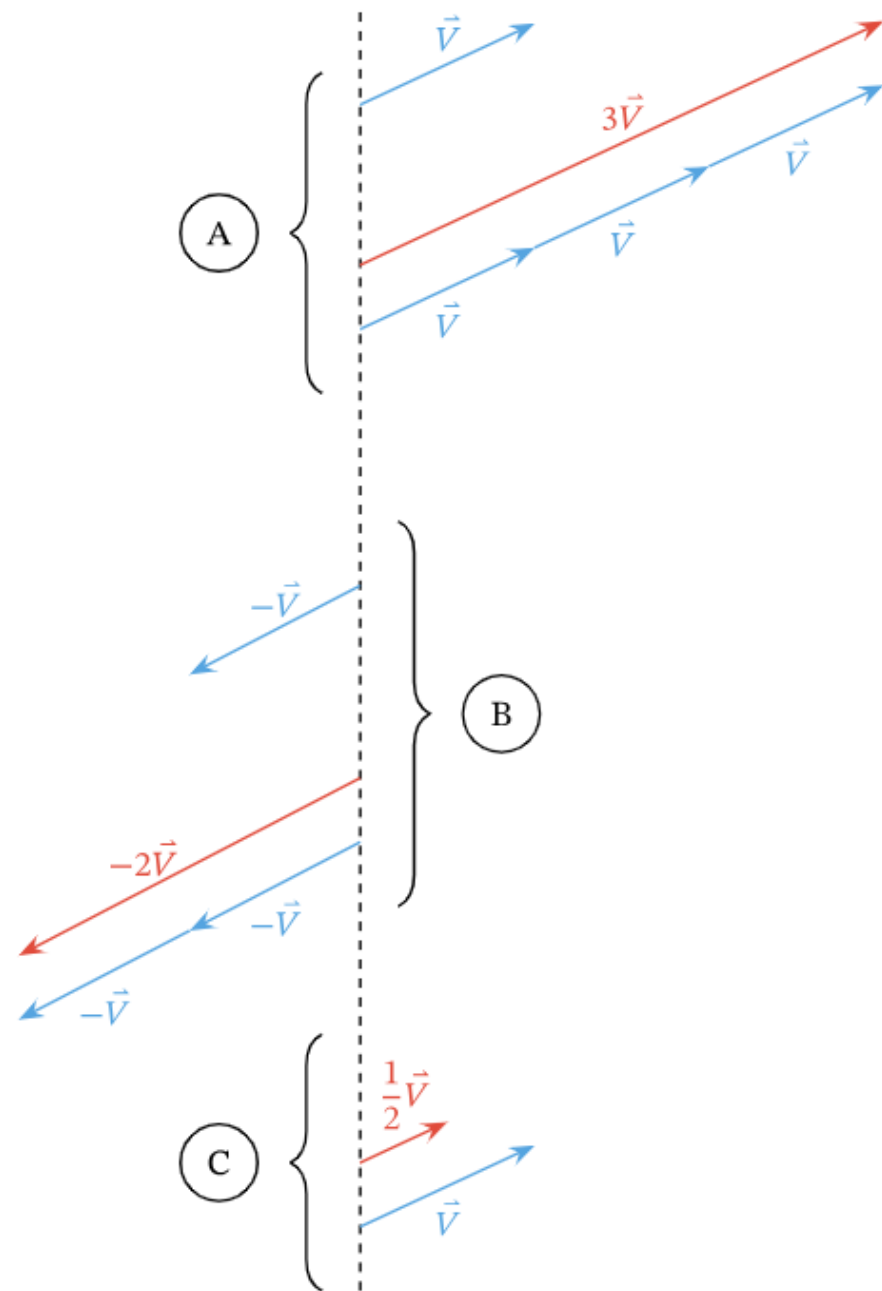
$$\Rightarrow \vec{p} + \vec{q} = \vec{q} + \vec{p}$$

خاصیت جایابی



$$a - b = a + (-b)$$





Dot product ضرب نقطه‌ای

$$a = \langle a_1, a_2, \dots, a_n \rangle$$

$$b = \langle b_1, b_2, \dots, b_n \rangle$$

$$a \cdot b = \underbrace{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}_{\text{مجموع}}$$

$$a \cdot a = |a|^2$$

$$a \cdot b = b \cdot a \quad \text{جابجایی}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$a = \langle 2, -1, 2 \rangle$$

$$b = \langle 1, -1, 0 \rangle$$

$$\left. \begin{array}{l} a = \langle 2, -1, 2 \rangle \\ b = \langle 1, -1, 0 \rangle \end{array} \right\} \text{زاویه بین دو بردار} = ?$$

$$|a| = \sqrt{9} = 3$$

$$|b| = \sqrt{2}$$

$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{2 + 1 + 0}{\sqrt{2} \times 3} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4} \text{ rad } 45^\circ$$

$$a: 2\vec{i} + 2\vec{j} - \vec{k}$$

$$b: 5\vec{i} - 4\vec{j} + 2\vec{k}$$

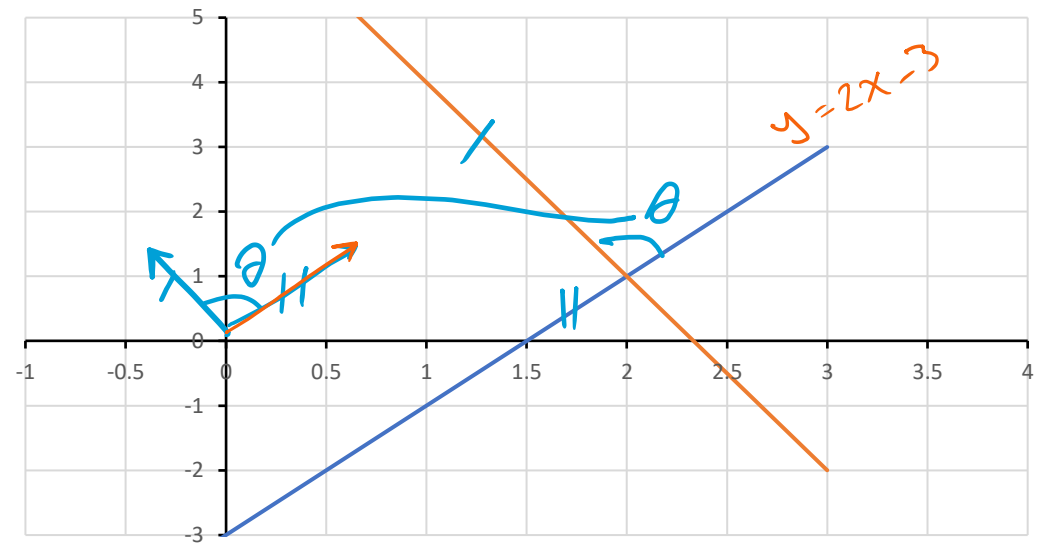
برهم عمودند

$$a \cdot b = 0 \Rightarrow 10 - 8 - 2 = 0$$

$$\begin{cases} 2x - y = 3 \\ y = 2x - 3 \end{cases} \quad a: \langle 1, 2 \rangle$$

$$\begin{cases} 3x + y = 7 \\ y = 7 - 3x \end{cases} \quad b: \langle 1, -3 \rangle$$

$$y = 7 - 3x$$

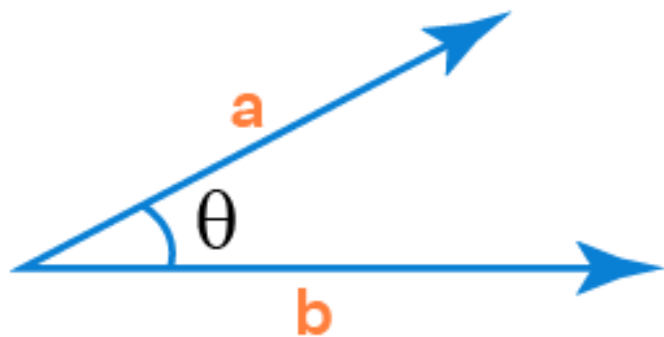


$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{1 - 6}{\sqrt{5} \times \sqrt{10}} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow$$

$$\theta = 45^\circ$$

$$\theta = 135^\circ$$

ویژگی‌های بردارها



$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

(OR)

$$\sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\text{ماتریس همانی} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{ماتریس مقدر} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 1 & 5 \end{bmatrix}$$

جمع و تفریق دو ماتریس
درایه به درایه

ضرب عدد در ماتریس

$$2 \times \begin{bmatrix} 3 & 2 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & -18 \end{bmatrix}$$

$$\underbrace{A}_{n \times k} \times \underbrace{B}_{k \times m} = \underbrace{C}_{n \times m}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1 \times 10 + 2 \times 20 + 3 \times 30 & 1 \times 11 + 2 \times 21 + 3 \times 31 \\ 4 \times 10 + 5 \times 20 + 6 \times 30 & 4 \times 11 + 5 \times 21 + 6 \times 31 \end{bmatrix}_{2 \times 2}$$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \\ 6 & 8 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & 8 \\ 5 & 6 & 7 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(A \times B)^T = B^T \times A^T \neq A^T \times B^T$$

$$A = A^T \quad * \text{ماتريس متقارن}$$

$$\text{مجموع مقادير قطر اعلی} \leftarrow \text{tr}(A) = \sum_{i=1}^n a_{ii} \quad * \text{انتر ماتريس}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\text{tr}(A) = 1 + 5 + 9 = 15$$

$$\|A\|_F = \sqrt{\sum a_{ij}^2} = \sqrt{\text{Tr}(AA^T)}$$

* نرم ماتریس

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = |A| = ad - bc$$

* دترمینان ماتریس

$$\det \begin{bmatrix} 8 & 3 \\ 4 & 2 \end{bmatrix} = \frac{8 \times 2}{16} - \frac{3 \times 4}{12} = 4$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$\text{sgn} = (-1)^{i+j}$$

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

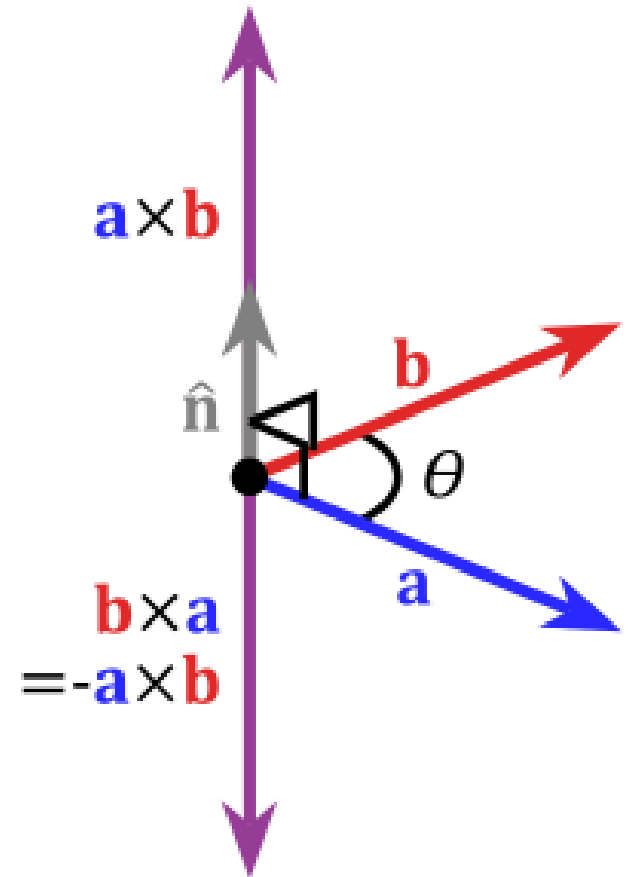
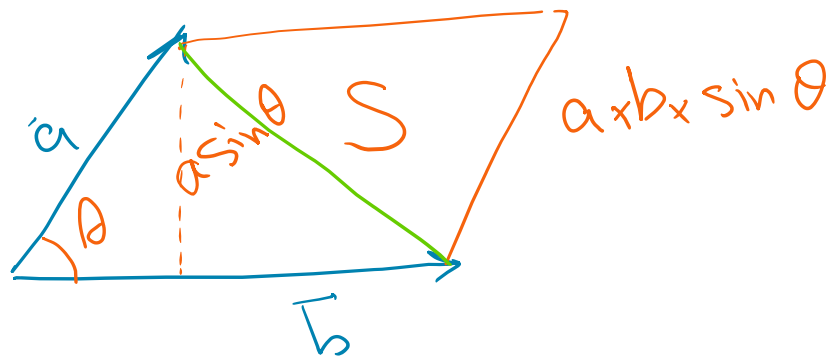
$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

$$a = \langle a_1, a_2, a_3 \rangle \quad a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$b = \langle b_1, b_2, b_3 \rangle \quad b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$|a \times b| = |a||b| \sin \theta$$



$$A(0,0,0)$$

$$B(1,0,2) \Rightarrow \text{مساحت مثلث؟}$$

$$C(2,-2,0)$$

$$A \rightarrow A^{-1}$$

$$A \cdot A^{-1} = I$$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1} \times A^{-1}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{8-9} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_{3 \times 3}^{-1} = \frac{1}{|A|} [Adjoint]$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^T =$$

$$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \quad (-1)^{i+j}$$

$$adj(A) = \begin{bmatrix} + | e h | - | b h | + | b e | \\ + | f i | - | c i | + | c f | \\ - | d g | + | a g | - | a d | \\ - | f i | + | c i | - | c f | \\ + | d g | - | a g | + | a d | \\ + | e h | - | b h | + | b e | \end{bmatrix}$$

ماتریس وارون

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

وارون ماتریس قطری

$$A X = \lambda X \quad \begin{array}{l} \text{مقدار ویژه} \downarrow \\ \text{بیدار ویژه} \end{array}$$

$$A X - \lambda X = 0$$

$$(A - \lambda I) X = 0 \quad \leftarrow$$

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

بیدار ویژه

مقدار ویژه

مقادیر ویژه و بیدار ویژه

$$a = (1, 1)$$

$$a = (a_1, a_2)$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \Rightarrow \begin{array}{c} A \\ \hline \end{array} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{array}{c} I \\ \hline \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

$$(1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda \left\{ \begin{array}{l} -2 \\ 5 \end{array} \right. \Rightarrow x_1 = x_2 \Rightarrow \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-(-2)}{3} & 4 \\ 3 & \frac{2-(-2)}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} 3x_1 + 4x_2 = 0 \Rightarrow x_1 = -\frac{4}{3}x_2 \\ 3x_1 + 4x_2 = 0 \end{array}$$

$$\begin{cases} -4x_1 + 4x_2 = 0 \Rightarrow x_1 = x_2 \\ 3x_1 - 3x_2 = 0 \Rightarrow x_1 = x_2 \end{cases}$$

$$\begin{bmatrix} -\frac{4}{3}a \\ a \end{bmatrix}$$

دستگاه معادلات خطی و ماتریس

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$A \times X = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A \underline{X} = b$$

$$\underbrace{A^{-1} \times A}_{I} \times X = A^{-1}b \Rightarrow X = A^{-1}b$$

$$\begin{cases} x_1 - x_2 = -1 \Rightarrow x_1 = -1 + x_2 & \xrightarrow[\text{در معادله}]{\text{قادر با این}} \\ 3x_1 + x_2 = 9 \end{cases} \quad 3(-1 + x_2) + x_2 = -1 \Rightarrow \begin{cases} x_2 = 3 \\ x_1 = 2 \end{cases}$$

$$AX = b$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} [\text{adj}(A)] = \frac{1}{1 - (-3)} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}^{-1}}_{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}} \begin{bmatrix} -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

روش کرامر

$$x_1 = \frac{\begin{vmatrix} -1 & -1 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{-1 - (-9)}{1 - (-3)} = \frac{8}{4} = 2$$

$$x_2 = \frac{\begin{vmatrix} 1 & -1 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{9 - (-3)}{4} = \frac{12}{4} = 3$$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 1 & 9 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 4 & 0 & 8 \\ 3 & 1 & 9 \end{array} \right] \xrightarrow{\frac{R_1}{4}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & 1 & 9 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array} \right] \begin{matrix} x_1 \\ x_2 \end{matrix}$$

I باء

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 5 \\ 3x_1 + 5x_2 + 6x_3 = 7 \\ 2x_1 + 4x_2 + 3x_3 = 8 \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}} = -15 \quad x_2 = 8 \quad x_3 = 2$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix} = 1(5 \times 3 - 6 \times 4) - 3(3 \times 3 - 6 \times 2) + (-2)(3 \times 4 - 5 \times 2)$$

برای هر چه مقدار از a دستگاه زیر بیشترین جواب دارد

$$\begin{cases} x - y + az = 0 \\ 3x - y + 2z = 0 \\ 6x - 4y + 5z = 0 \end{cases}$$

$$AX = 0$$

$$|A| = 0 \Rightarrow \text{بیشترین جواب}$$

(نقطه شروع) P_0
 $\vec{v} \langle a, b, c \rangle$

$$y - y_0 = m(x - x_0)$$

معادلات خط و صفحه در مقاربت به هم

$$\vec{r} = \vec{r}_0 + \vec{a}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

معادله بردار خط

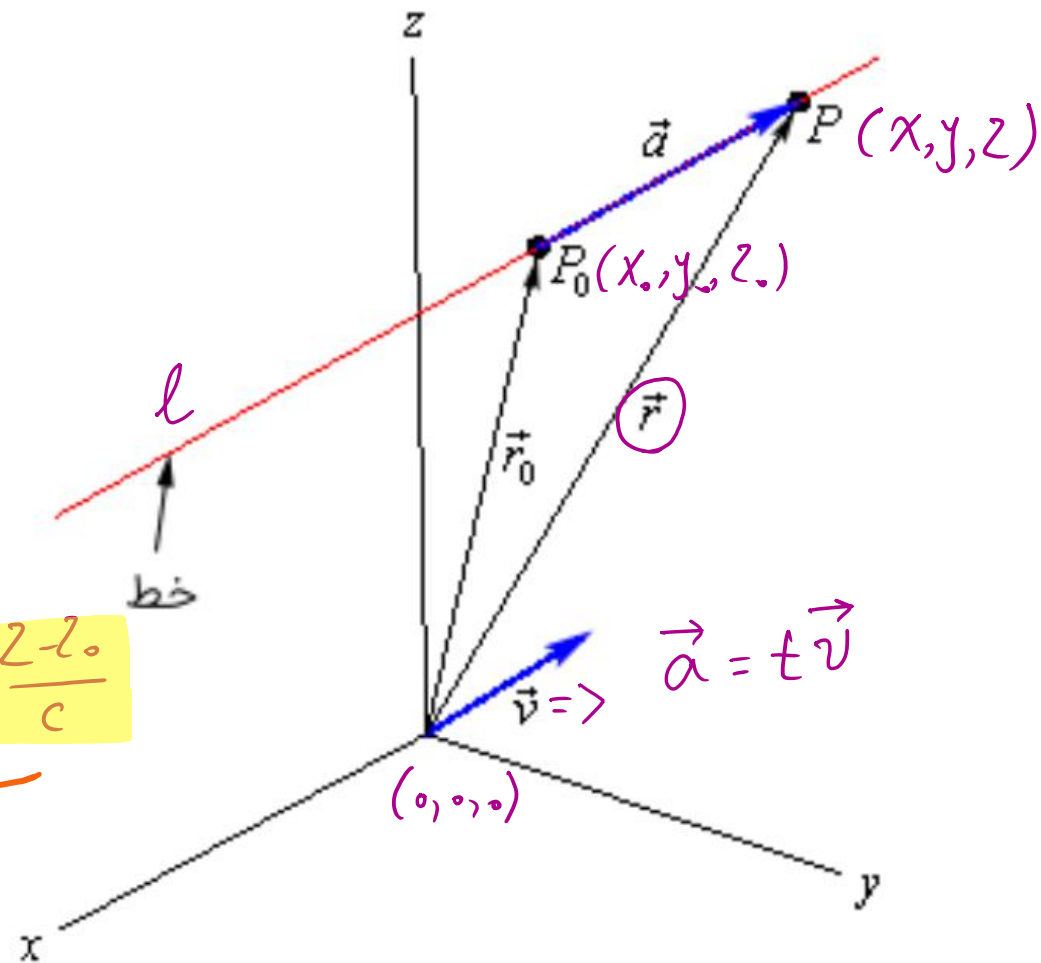
$$(x_0 + ta)\vec{i} + (y_0 + tb)\vec{j} + (z_0 + tc)\vec{k}$$

معادله پارامتر خط

$$\left\{ \begin{array}{l} x = x_0 + ta \Rightarrow t = \frac{x - x_0}{a} \\ y = y_0 + tb \Rightarrow t = \frac{y - y_0}{b} \\ z = z_0 + tc \Rightarrow t = \frac{z - z_0}{c} \end{array} \right.$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

معادلات متقاربه خط



مثال) $P_0(2, 2.4, 3.5)$

$$\vec{a} = \underbrace{3\vec{i}}_a + \underbrace{2\vec{j}}_b - \underbrace{1\vec{k}}_c$$

$$r = r_0 + t v$$

$$\langle x, y, z \rangle = \langle x_0 + t a, y_0 + t b, z_0 + t c \rangle$$

$$\langle 2 + 3t, 2.4 + 2t, 3.5 + (-1)t \rangle$$

$$\text{معالله بردار} (2+3t)\vec{i} + (2.4+2t)\vec{j} + (3.5-t)\vec{k}$$

$$\vec{a} \begin{cases} A(-4, -6, 1) \\ B(-2, 0, -3) \end{cases} \rightarrow \vec{b} \begin{cases} C(10, 18, 4) \\ D(5, 3, 14) \end{cases} \Rightarrow$$

$$\vec{a} = \langle 2, 6, -4 \rangle$$

$$\vec{b} = \langle -5, -15, 10 \rangle$$

$$\frac{b_1}{a_1} = \frac{-5}{2}$$

$$\frac{b_2}{a_2} = \frac{-15}{6} = \frac{-5}{2}$$

$$\frac{b_3}{a_3} = \frac{10}{-4} = \frac{-5}{2}$$

$$\vec{b} = \left(\frac{-5}{2} \right) \vec{a}$$

این دو بردار موازی هستند

$$\left. \begin{array}{l} \ell_1: x=3+2t \quad y=4-t \quad z=1+3t \\ \ell_2: x=1+4s \quad y=3-2s \quad z=4+5s \end{array} \right\} \Rightarrow \begin{array}{l} v_1 = \langle 2, -1, 3 \rangle \\ v_2 = \langle 4, -2, 5 \rangle \end{array}$$

$v_1 \nparallel v_2$
موازی نیستند

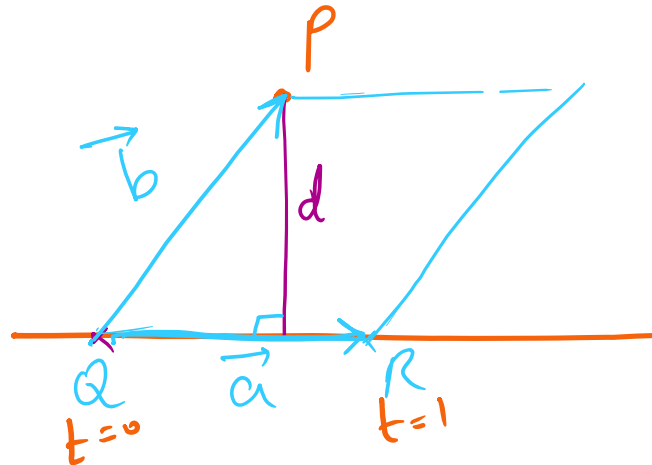
$$P(4, 1, -2)$$

فاصله نقطه P از خط ℓ_1

$$x=1+t$$

$$y=3-2t$$

$$z=4-3t$$



$$|a \times b| = d \times |a| \Rightarrow d = \frac{|a \times b|}{|a|}$$

$$t=0 \Rightarrow Q = \langle 1, 3, 4 \rangle$$

$$t=1 \Rightarrow R = \langle 2, 1, 1 \rangle$$

$$\vec{b} = \langle 3, -2, -6 \rangle$$

$$\vec{a} = \langle 1, -2, -3 \rangle$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix} = \underbrace{(12-6)}_6 \vec{i} - \underbrace{(-6+9)}_{-3} \vec{j} + \underbrace{(-2+6)}_4 \vec{k}$$

$$* \Rightarrow d = \frac{\sqrt{36+9+16}}{\sqrt{1+4+9}} = \frac{\sqrt{61}}{\sqrt{14}}$$

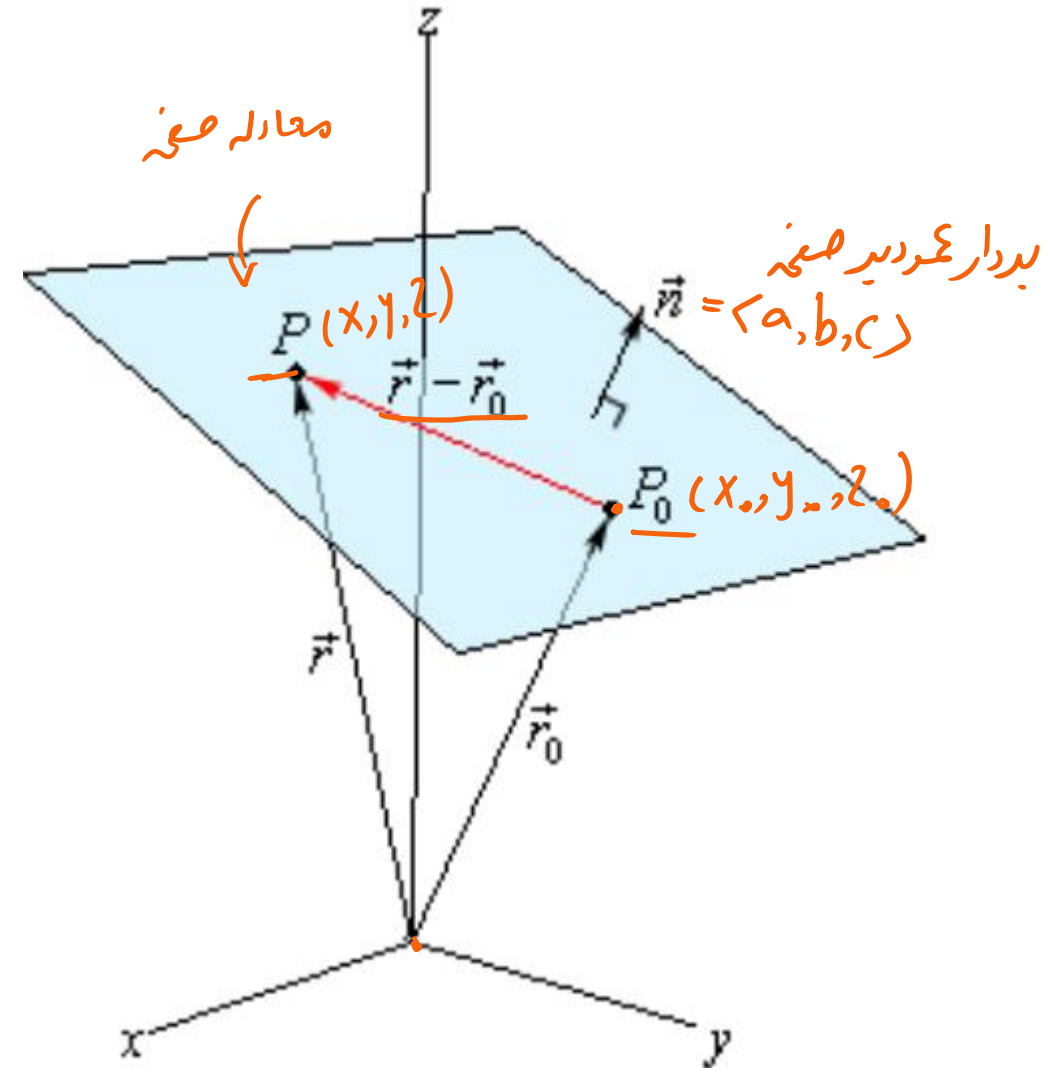
مقدار داخلی برابر با صفر است $\Rightarrow n \perp (r-r_0)$

$$\langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

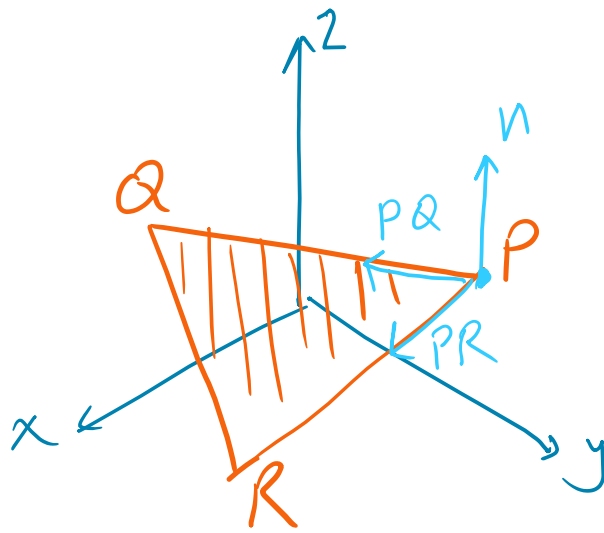
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz - \underbrace{ax_0 + by_0 + cz_0}_{d} = 0$$

$$\boxed{ax + by + cz - d = 0}$$



$$\begin{aligned} R & (5, 2, 0) \\ Q & (3, -1, 6) \\ P & (1, 3, 2) \end{aligned}$$



$$\vec{PQ} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

$$n = PQ \times PR = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \underbrace{12}_{a} \vec{i} + \underbrace{20}_{b} \vec{j} + \underbrace{14}_{c} \vec{k}$$

$$ax + by + cz - d = 0$$

$$12x + 20y + 14z - (12 \times 1 + 20 \times 3 + 14 \times 2) = 0$$

$$6x + 10y + 7z - 50 = 0$$

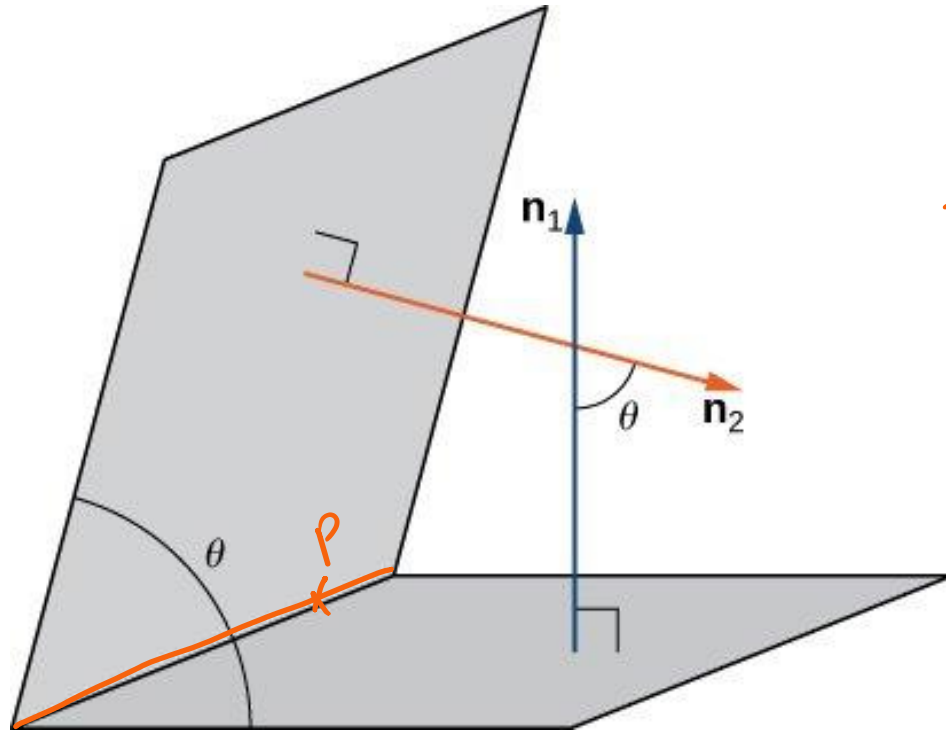
* تمرین

نقطه اشتراک خط زیر با صفحه $4x + 5y - 2z = 18$

$$\begin{cases} x = 2 + 3t \\ y = -4t \\ z = 5 + t \end{cases}$$

* معادله خط گذرنده از نقطه $(6, 0, 1)$ و عمود بر صفحه $x + 3y + 2z = 5$ را مشخص کنید

دو صفحه موازی و عمود و زاویه بین دو صفحه



1) $n_1 = 1 < n_2$ موازی
2) $n_1 \cdot n_2 = 0$ عمود

3) $n_1 \cdot n_2 = |n_1| \cdot |n_2| \times \cos \theta$

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|}$$

تمرین *

زاویه بین دو صفحه
مقادیر خط
استدک

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases}$$

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{42}} \right) \approx 72^\circ$$

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

مثال

$$\begin{cases} x + 4y - 3z = 1 \\ -3x + 6y + 7z = 0 \end{cases}$$

$n_1 = (1, 4, -3)$ $n_2 = (-3, 6, 7)$

$n_1 \cdot n_2 = 0$

فاصله نقطه از صفحه

برای اسکالر
 n و v

$$d = \frac{|n \cdot v|}{|n|}$$

$v = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

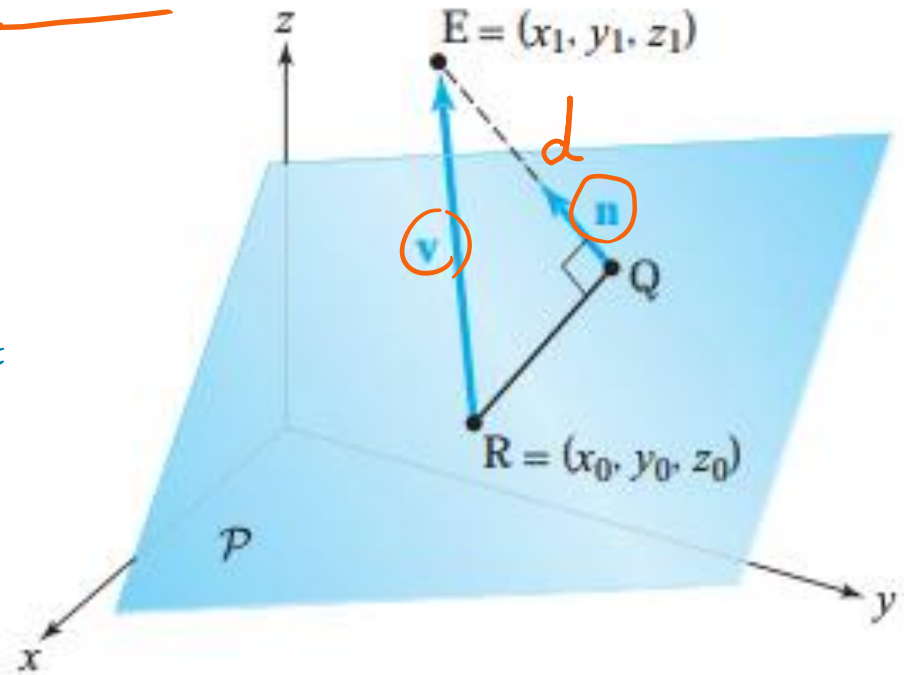
$$d = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

(a, b, c)

* مثال: فاصله نقطه $(1, -2, 4)$ از صفحه $3x + 2y + 6z = 5$

$n = \langle 3, 2, 6 \rangle$

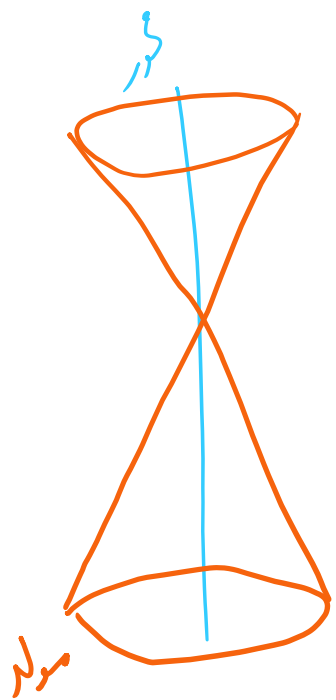
$$d = \frac{|3 \times 1 + 2(-2) + 6 \times 4 - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{18}{7}$$



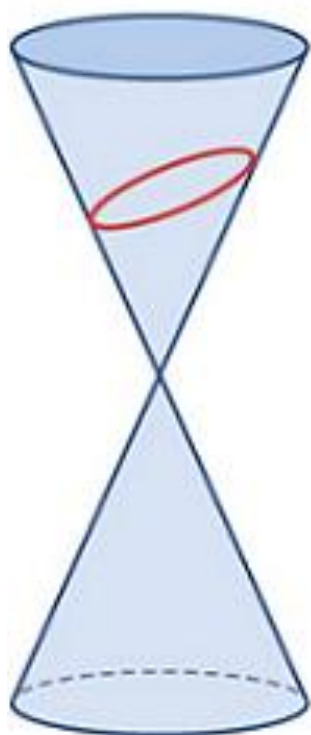
$$\begin{cases} \underline{a}x + \underline{b}y + cz + d_1 = 0 \\ \underline{a}x + \underline{b}y + cz + d_2 = 0 \end{cases} \Rightarrow d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\begin{cases} 10x + 2y - 2z = 5 \\ 5x + y - z = 1 \end{cases} \Rightarrow \text{فاصله این دو صفحه را بیابید}$$

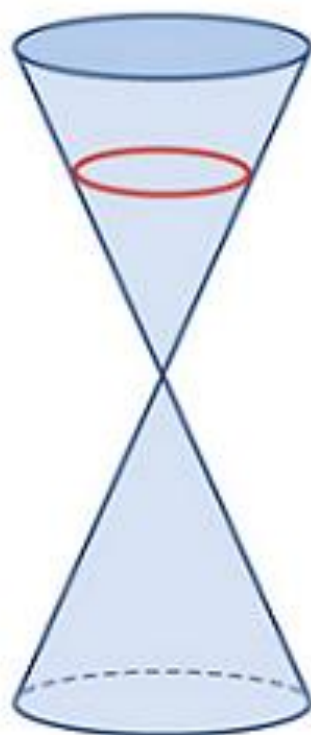
* تمرین



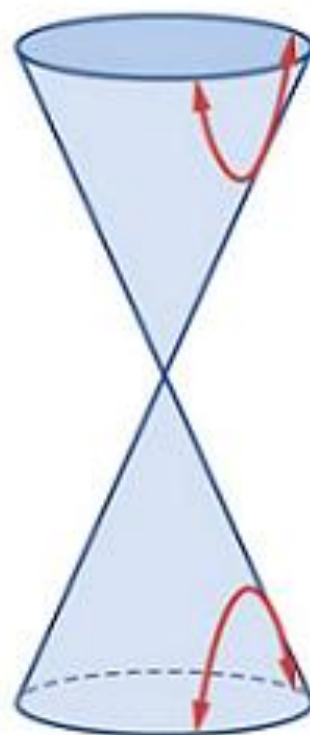
مولد



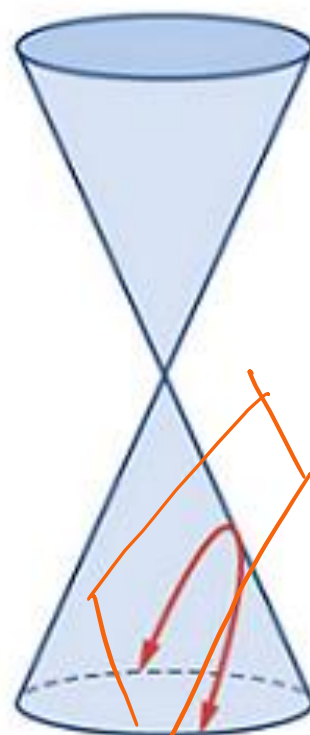
بیضی



دایره

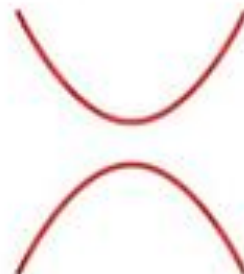
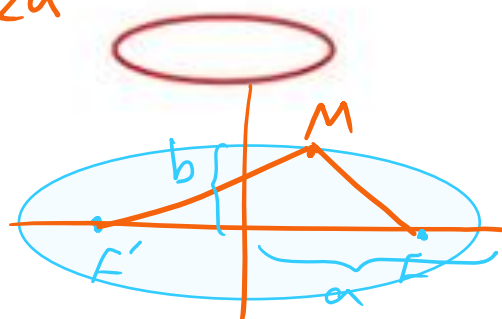


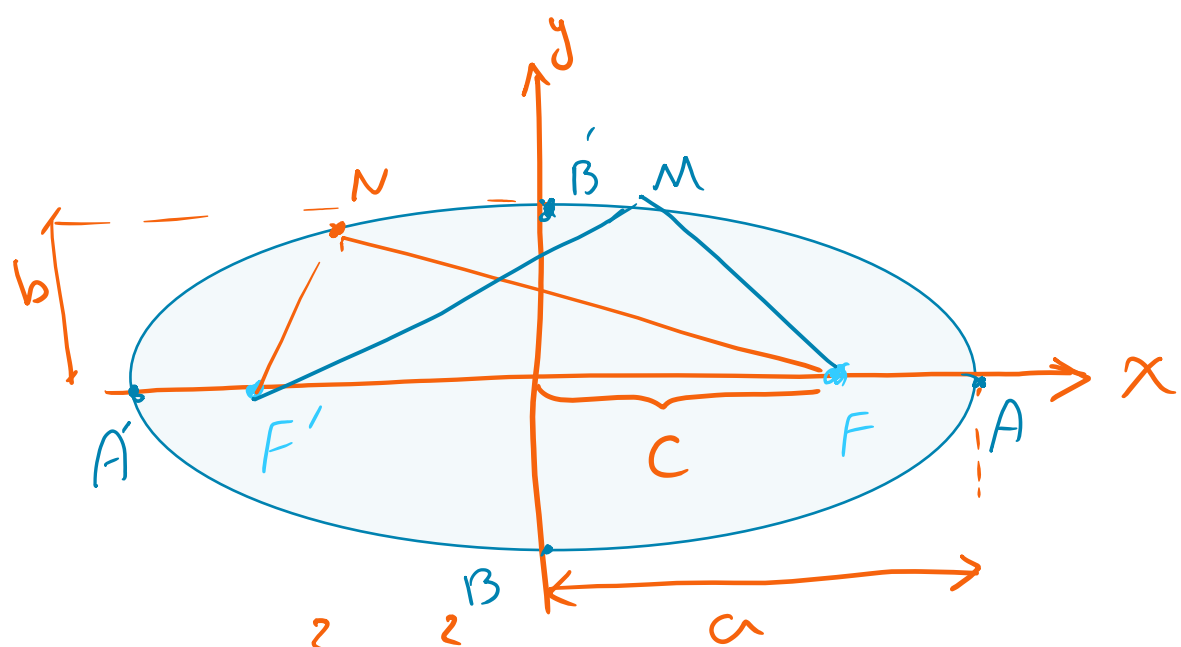
هذلولی



سهمی

$$|MF'| + |MF| = 2a$$





معادله بیضی

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$|MF'| + |MF| = 2a$$

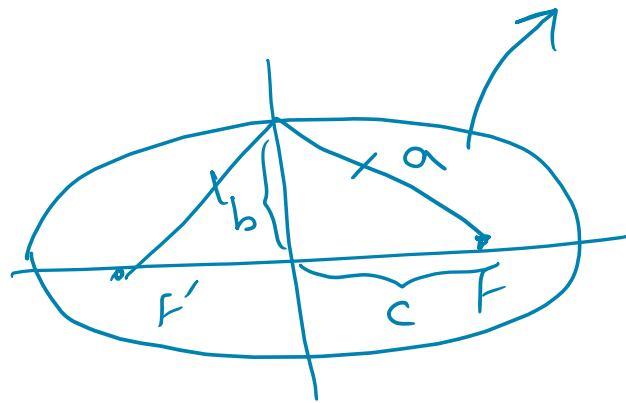
$$|NF'| + |NF| = 2a$$

$$|A'A| = 2a$$

$$|B'B| = 2b$$

$$|F'F| = 2c$$

$$a^2 = b^2 + c^2$$



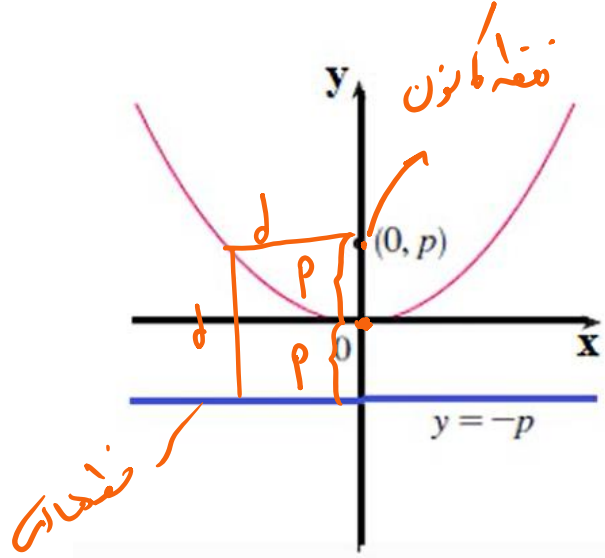
* تمرین

فاصله کانونی؟

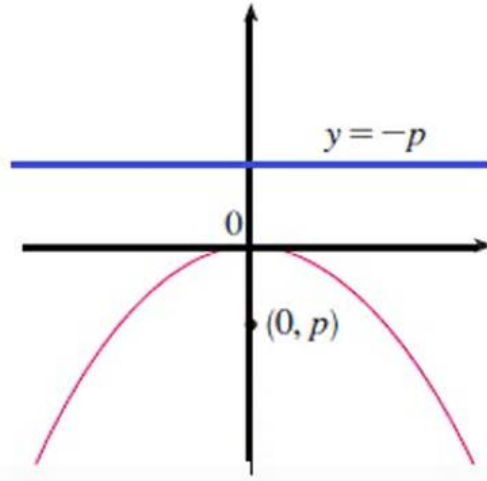
$$9x^2 + 16y^2 = 144$$

پارسه (Parabola)

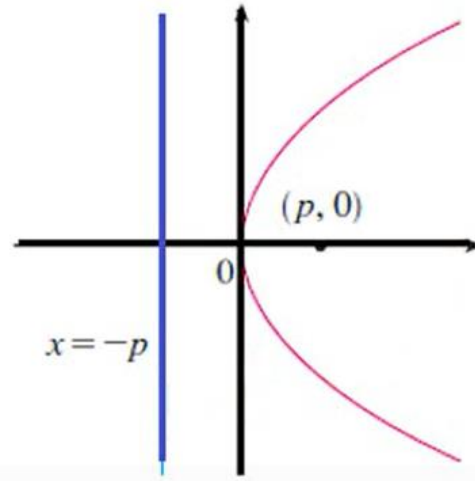
مکان هندسه نقاط که نامله آئنا از يك نطقه ثابت (كارزون) و يك خط ثابت (خط هاله) سطر به برابر است.



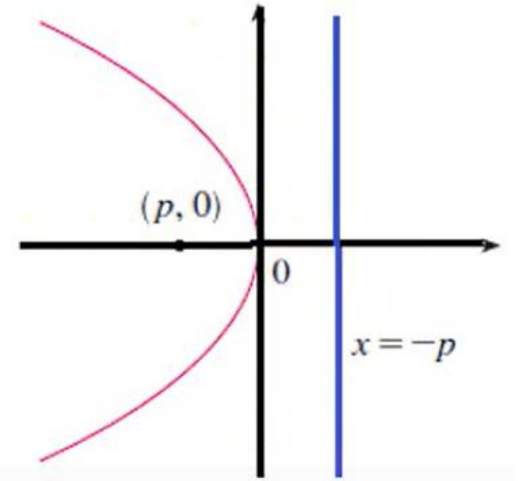
$$x^2 = 4py$$



$$x^2 = -4py$$



$$y^2 = 4px$$



$$y^2 = -4px$$

$$(y - \beta)^2 = 4p(x - \alpha)$$

(α, β) مختصات مركز