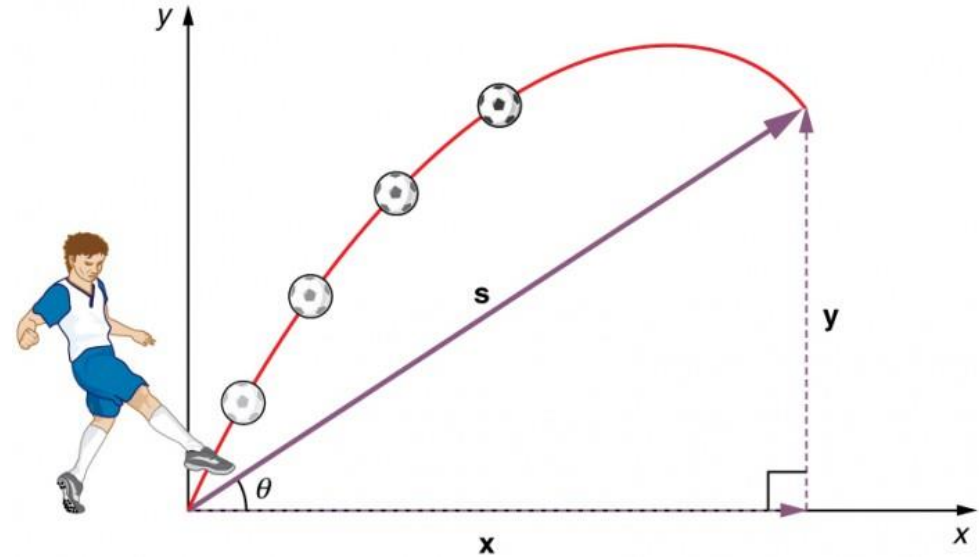
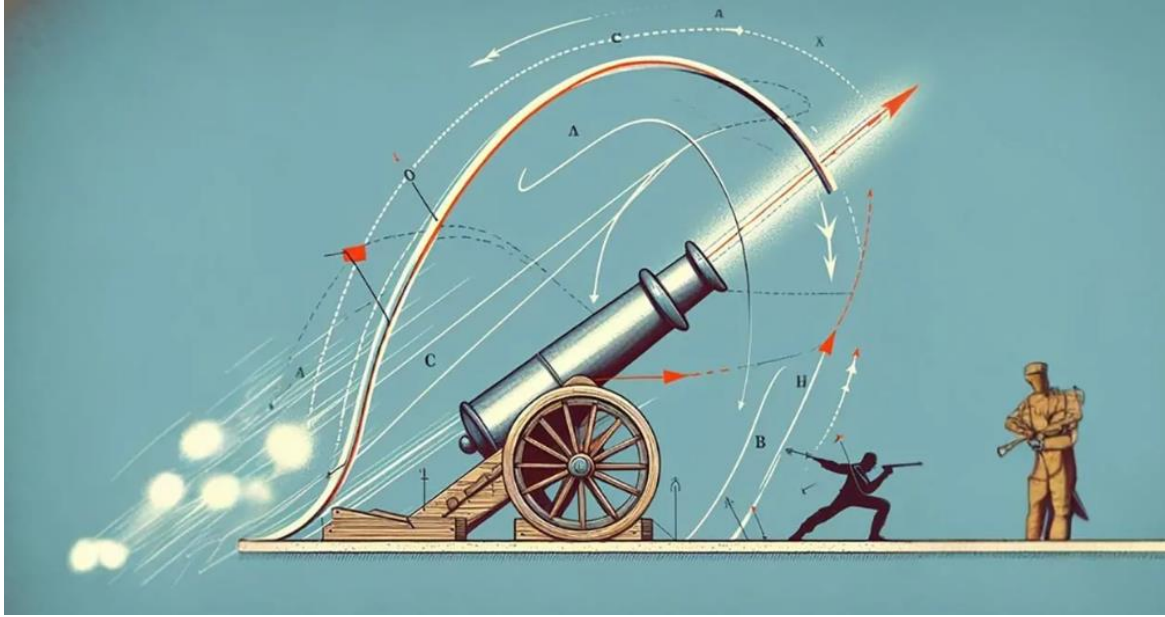




$$x(t) = V_0 t \cos(\theta)$$

$$y(t) = -\frac{1}{2}gt^2 + V_0 t \sin\theta$$

$$x = t^2 - t \quad \rightarrow \quad 0 = t^2 - t \Rightarrow t(t-1) = 0 \quad \left\{ \begin{array}{l} t=0 \\ t=1 \end{array} \right.$$

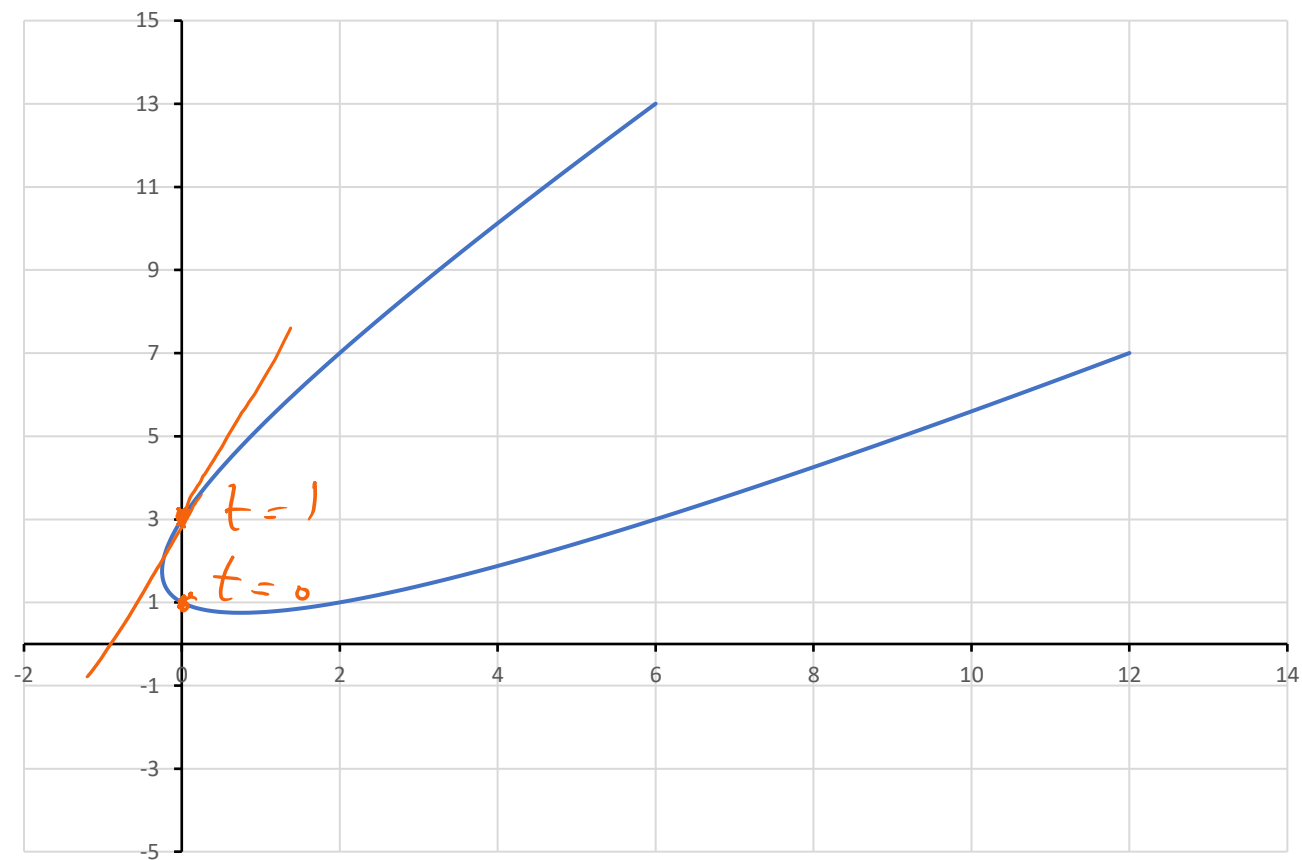
$$y - y_0 = m(x - x_0)$$

$$y = t^2 + t + 1$$

$$(x, y) = (0, 3)$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t-1} = \frac{3}{1} = 3$$

$$\underline{y = 3x + 3}$$



$$x = t^2 - 2t = 8$$

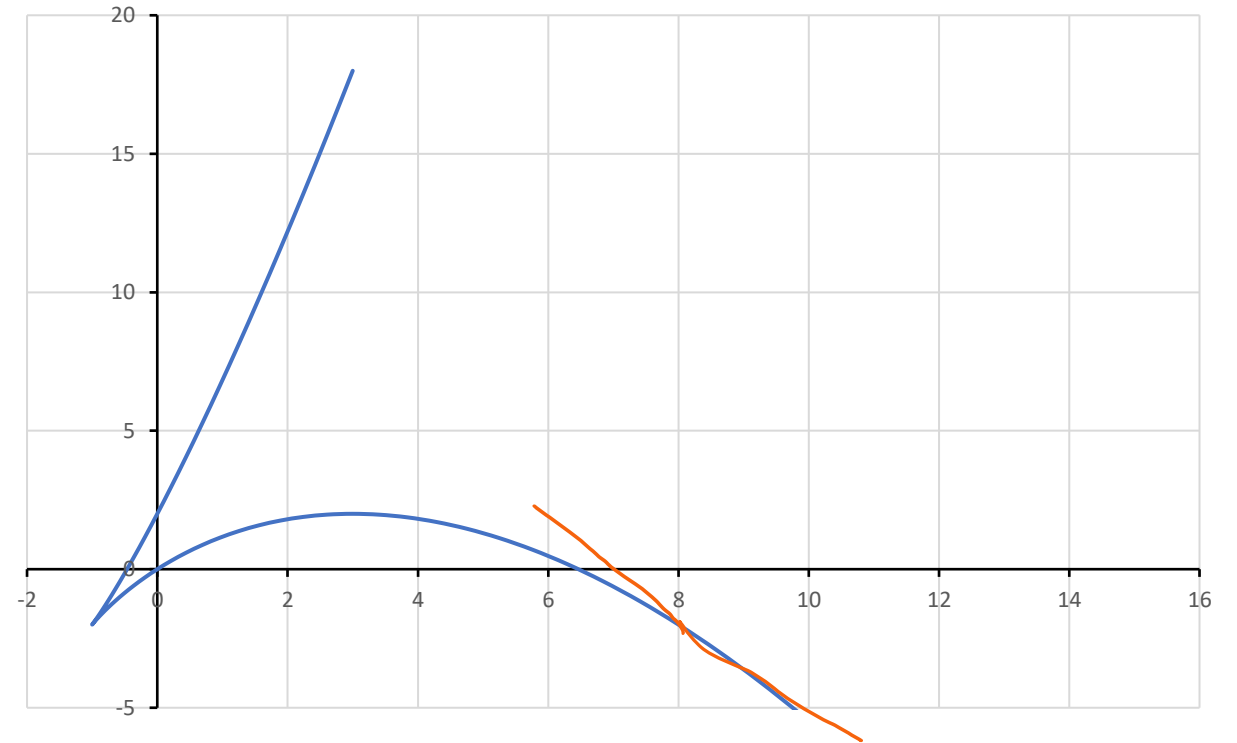
$$y - (-2) = m(x - 8)$$

$$y = t^3 - 3t \quad -8 + 6 = -2$$

$$t = -2$$

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t - 2} = \frac{12 - 3}{-6} = \frac{+9}{-6} = -\frac{3}{2}$$

$$y = -\frac{3}{2}(x - 8) - 2 = -\frac{3}{2}x + 10$$



$$x = t^2$$

$$y = t^3 - 3t$$

نقاط مفردة

$$\frac{dx}{dt} = 0, \frac{dy}{dt} \neq 0$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dx}{dt} \neq 0, \frac{dy}{dx} = 0$$

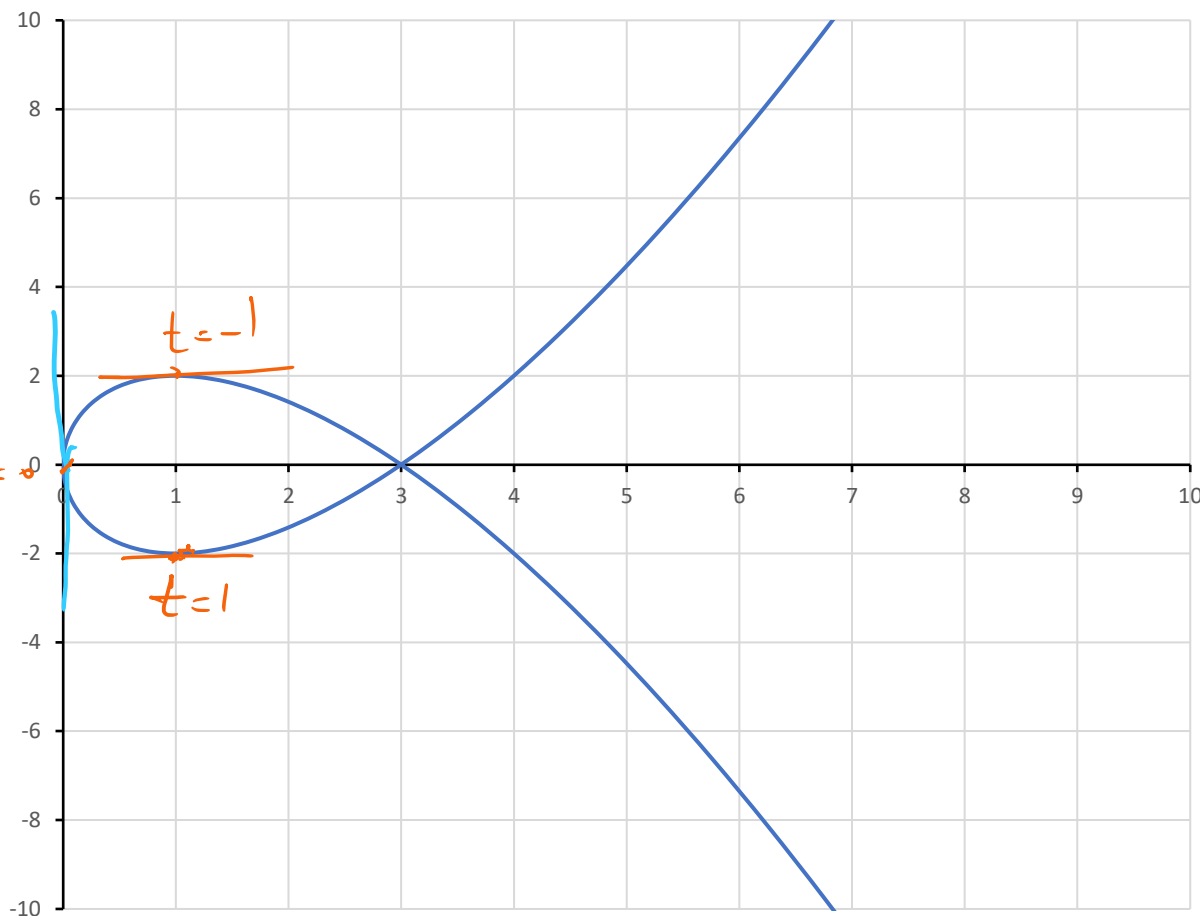
$t=0$

$$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2 - 1) \Rightarrow \begin{cases} t = -1 \\ t = 1 \end{cases}$$

$$\frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \Rightarrow y=0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow \begin{cases} t=0 \\ t=\pm\sqrt{3} \end{cases}$$

$$m = \frac{3t^2 - 3}{2t} \Rightarrow \frac{6}{2\sqrt{3}} = \sqrt{3}, -\sqrt{3}$$



$$t = \sqrt{3} \Rightarrow x = 3, y = (\sqrt{3}^3 - 3\sqrt{3})$$

$$x = r(\theta - \sin(\theta))$$

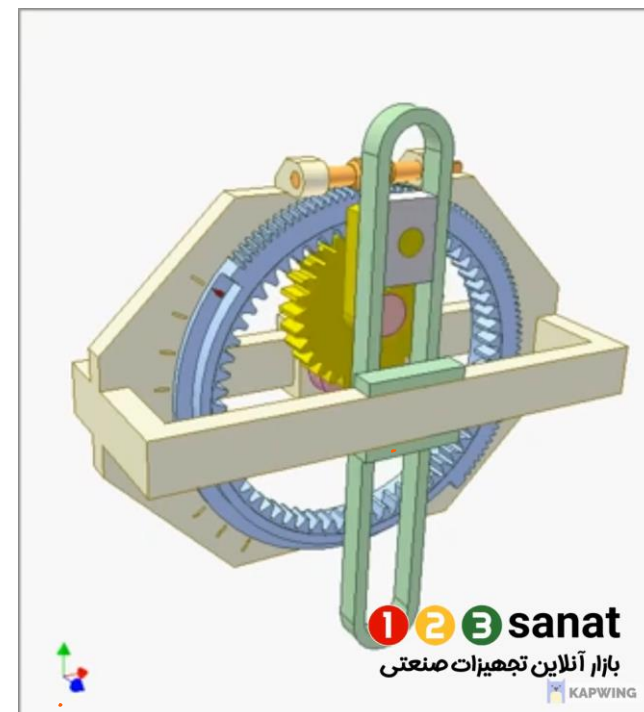
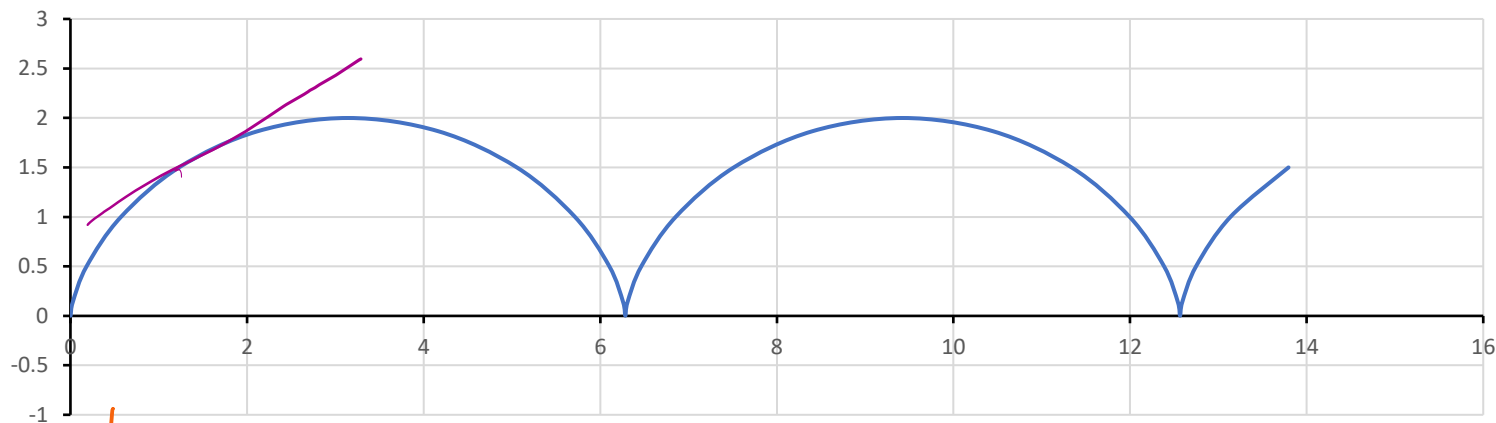
$$y = r(1 - \cos(\theta))$$

-sin θ

$$\theta = \frac{\pi}{3}$$

$$m = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\sin\theta)}{r(1 - \cos\theta)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \quad x, y$$

$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \quad y = r\left(\frac{1}{2}\right) \frac{r}{2} \quad y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$



$$x = t^2 + 1$$

$$y = t^2 + t$$

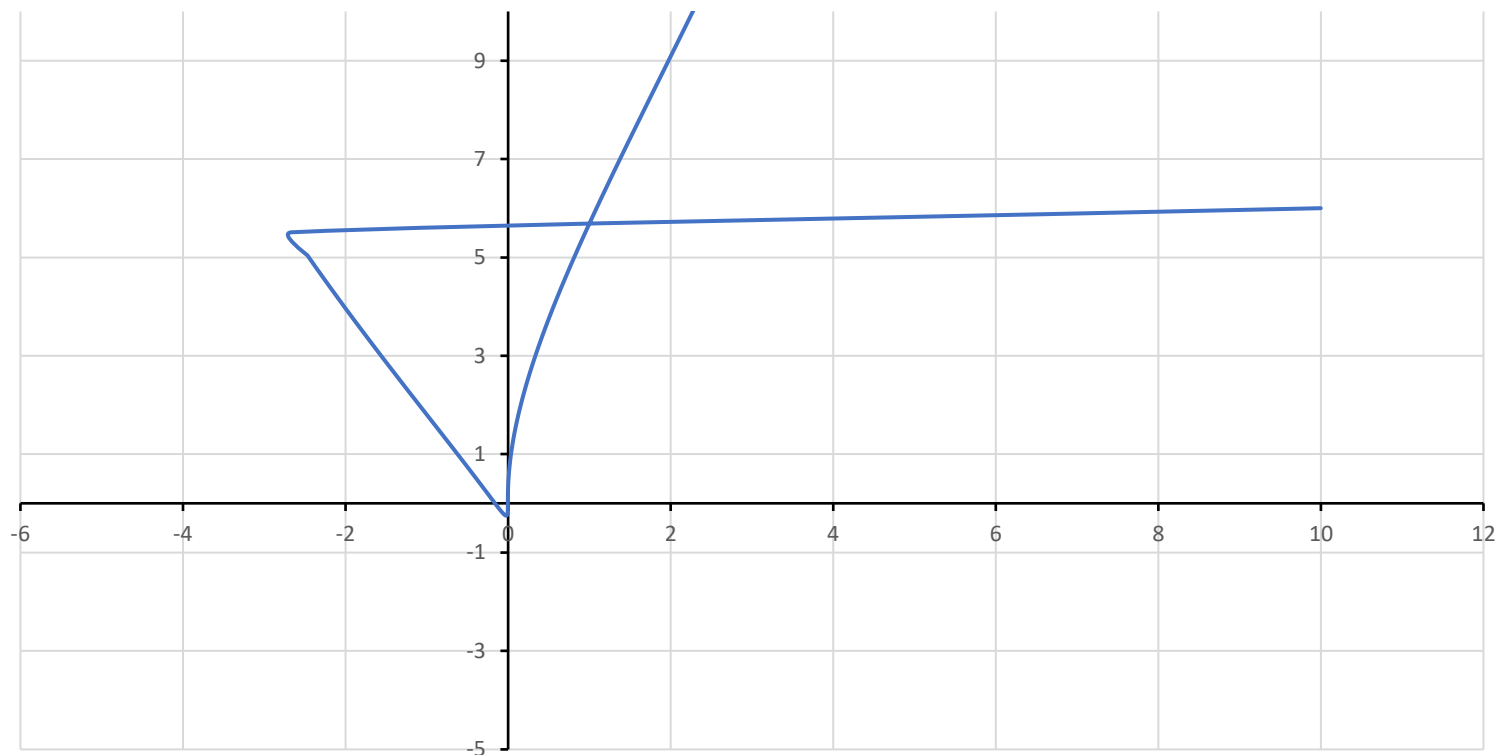
$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t}$$

$$\frac{d}{dt} \left(\frac{\frac{dy}{dx}}{\frac{dx}{dt}} \right)$$

$$\frac{4t - 4t}{2(2t)^2 - 2(2t+1)}$$

$$\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{-2}{8t^3} = \frac{-1}{4t^3} \Rightarrow \text{منفرد}$$



$$x = t^2$$

$$y = t^3 - 3t$$

$$\frac{d^2 y}{dx^2} = ? \quad \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

$$\frac{d}{dt} \rightarrow$$

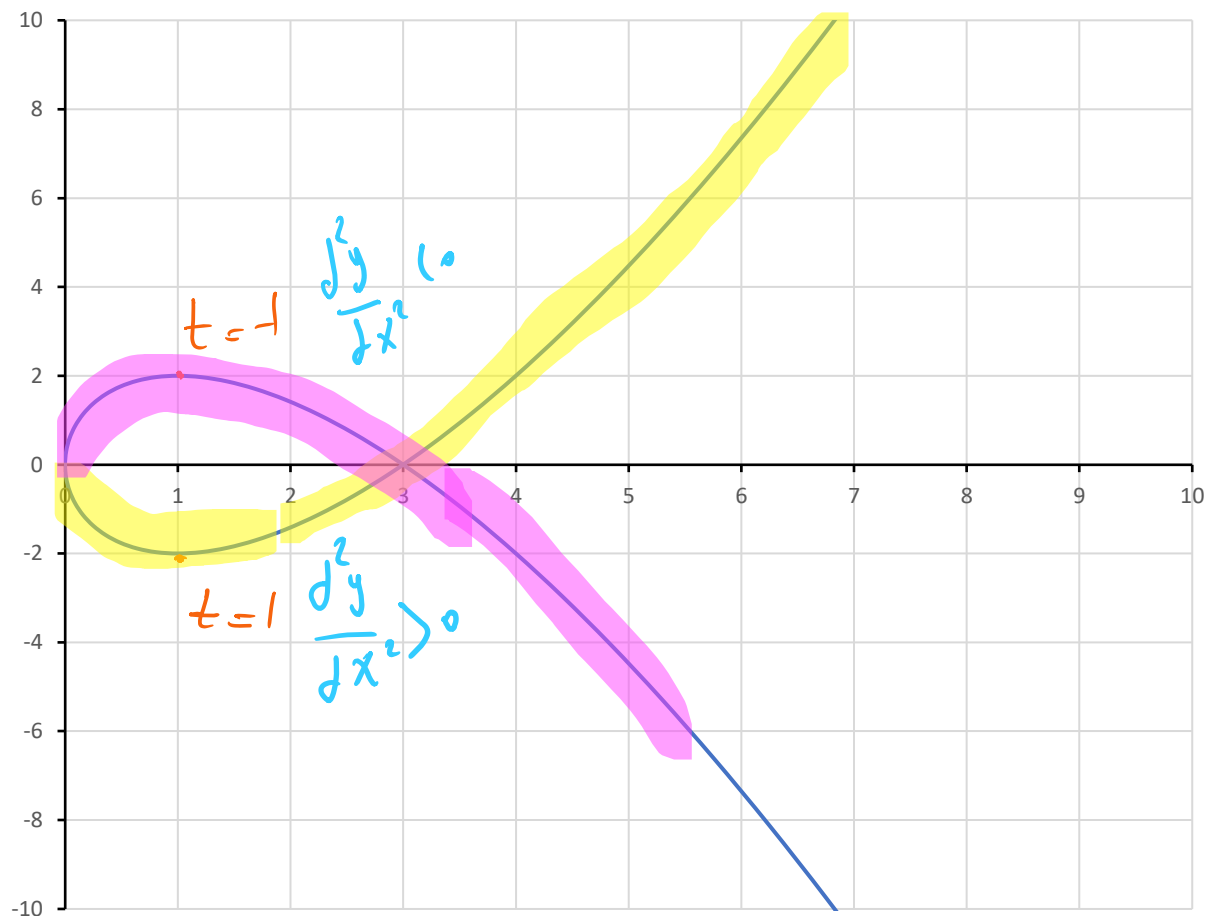
$$\frac{12t^2 - 6t^2 + 6}{4t^2}$$

$$= \frac{6t^2 + 6}{4t^2}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{6t^2 + 6}{4t^2}$$

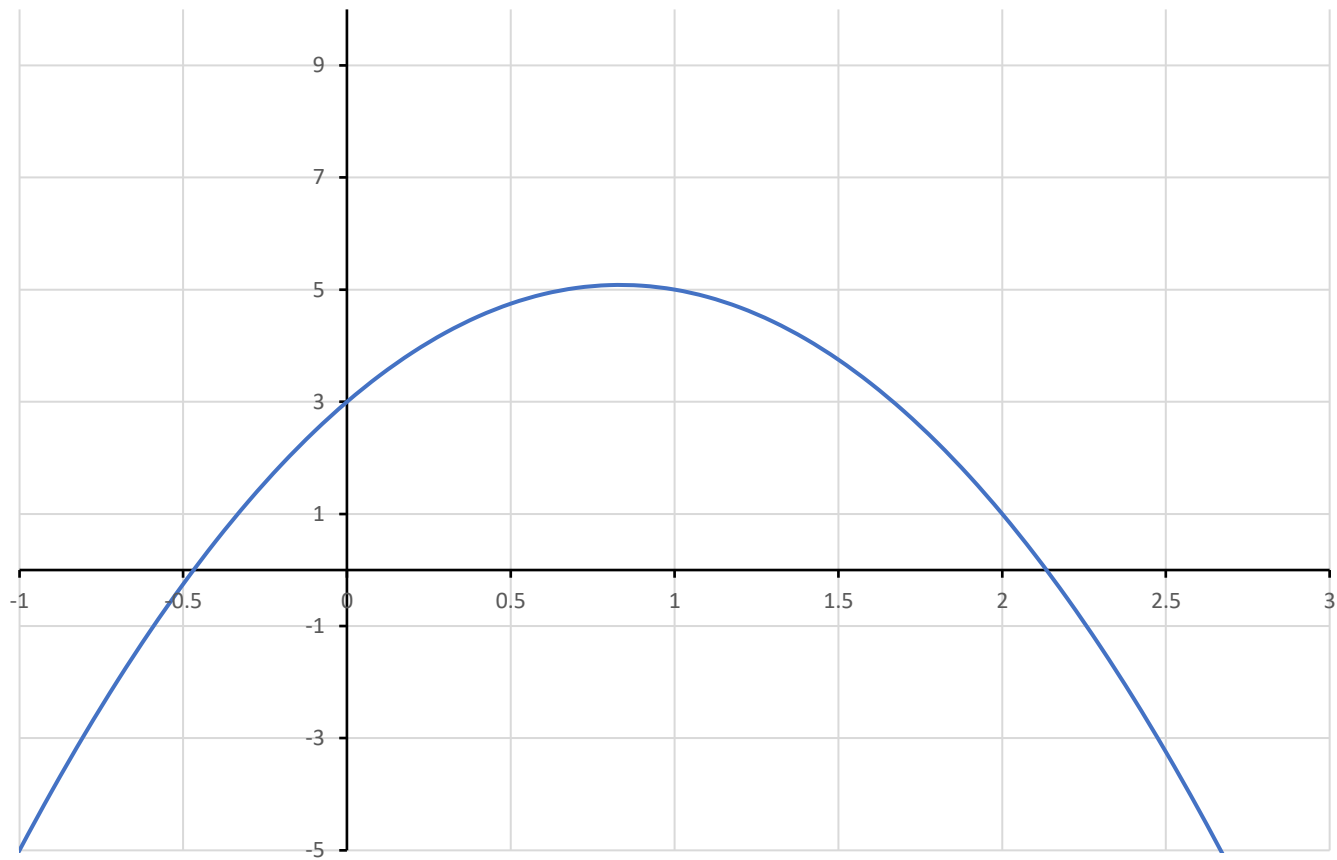
$$\frac{d^2 y}{dx^2} = \frac{\frac{6t^2 + 6}{4t^2}}{2t} = \frac{6(t^2 + 1)}{8t^3} = \frac{3(t^2 + 1)}{4t^3}$$



$$A = \int_a^b y \, dx = \int_\alpha^\beta g(t)f'(t)dt$$

$$x = f(t) \Rightarrow dx = f'(t)dt$$

$$y = g(t)$$



$$A = \int_a^b y \, dx = \int_\alpha^\beta \overbrace{g(t)}^{f'(t)} f'(t) \, dt$$

$$x = t^3 + 1 \Rightarrow dx = 3t^2 dt \quad 0 < t < 2$$

$$y = 2t - t^2$$

$$A = \int_0^2 (2t - t^2) 3t^2 \, dt = 3 \int_0^2 (2t^3 - t^4) \, dt$$

$$= 3 \left[\frac{2t^4}{4} - \frac{t^5}{5} \right]_0^2 = \frac{24}{5}$$



$$x = r(\theta - \sin(\theta))$$

$$y = r(1 - \cos(\theta))$$

$$dx = r(1 - \cos \theta) d\theta$$

$$0 < \theta < 2\pi$$

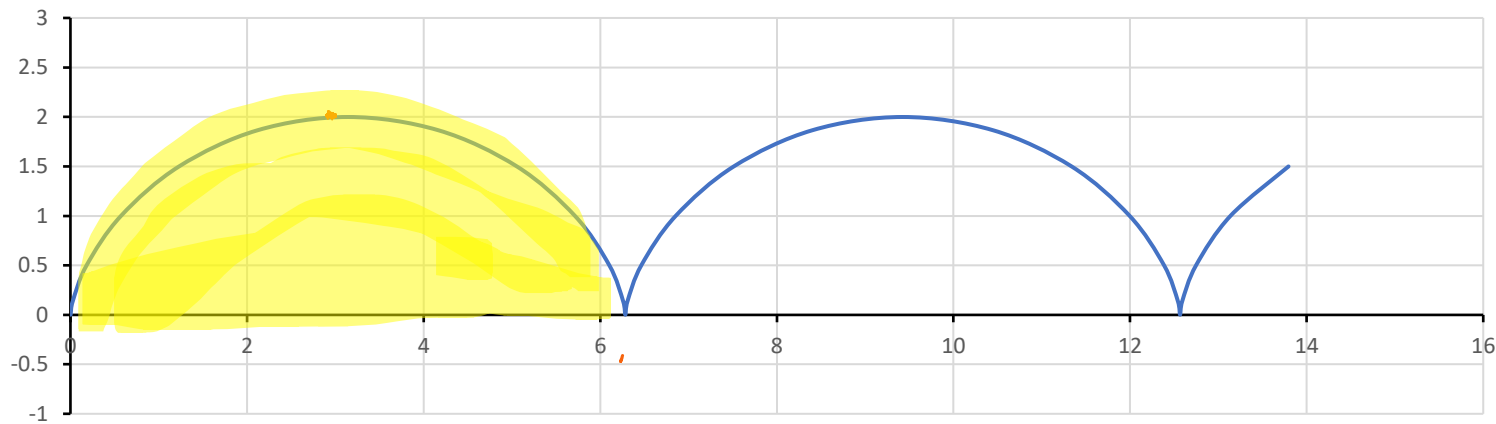
$$A = \int y dx = \int_0^{2\pi} g(\theta) f'(\theta) d\theta$$

$$A = \int r^2 (1 - \cos \theta)(1 - \cos \theta) d\theta$$

$$A = r^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = r^2 \int_0^{2\pi} (1 + \cos^2 \theta - 2\cos \theta) d\theta = r^2 \left[\frac{3}{2} \theta + \frac{1}{2} \left(\frac{1}{2} \sin 2\theta \right) - 2\sin \theta \right]_0^{2\pi}$$

$$r^2 \int_0^{2\pi} \left[\frac{3}{2} + \frac{1}{2} \cos 2\theta - 2\cos \theta \right] d\theta$$

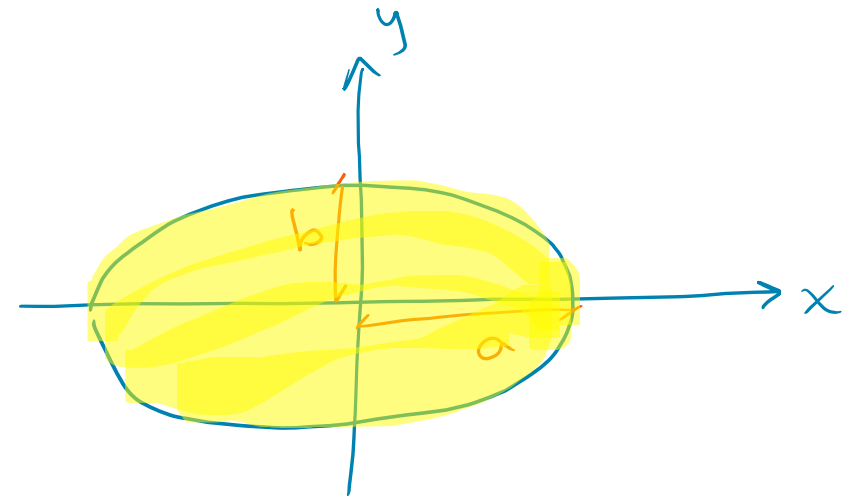
$$= r^2 \left[\frac{3}{2} (2\pi) \right] = \underline{3\pi r^2}$$



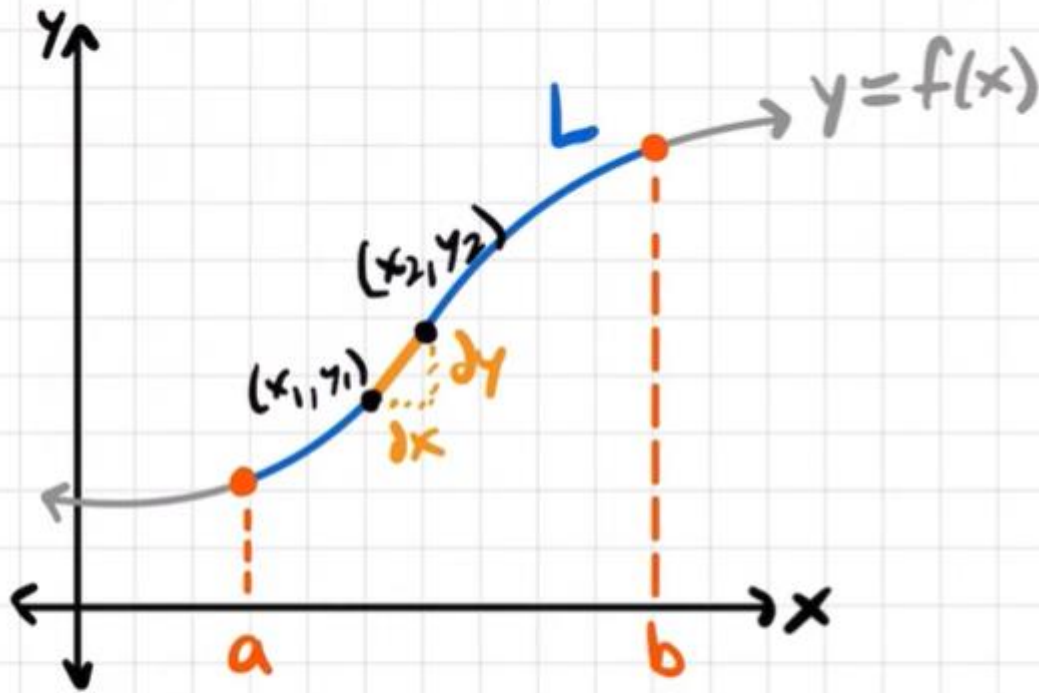
$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned}$$

$$\left. \begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \right\} \begin{aligned} \frac{x}{a} &= \cos \theta \\ \frac{y}{b} &= \sin \theta \end{aligned}$$

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$



Lesson 6: Arc Length



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b \sqrt{(dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2}\right)}$$

$$= \int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

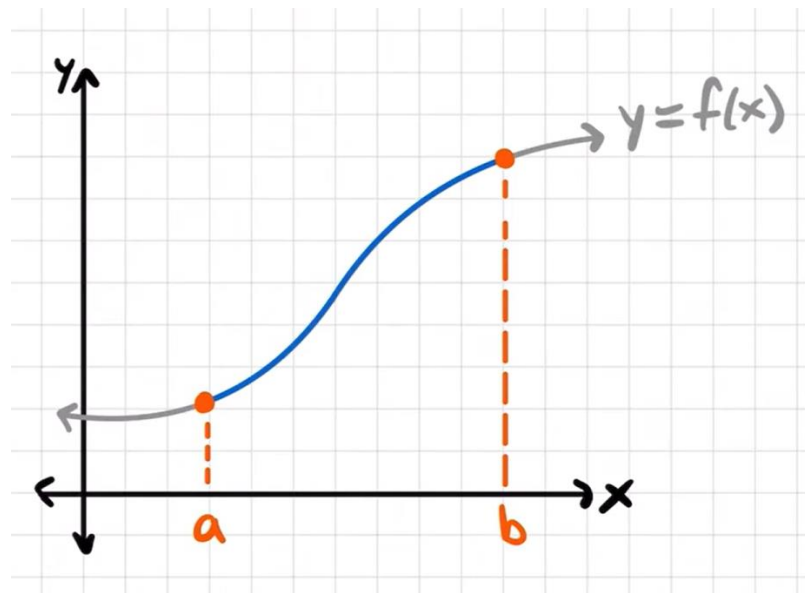
$$= \int_a^b \sqrt{1 + (f'(x))^2} dx$$



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$x = f(t)$$

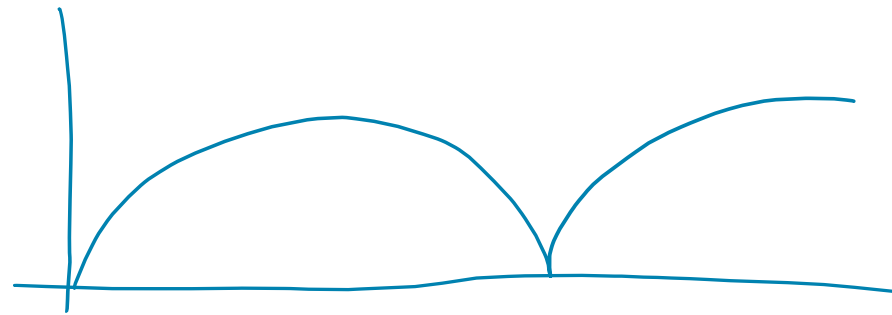
$$y = g(t)$$

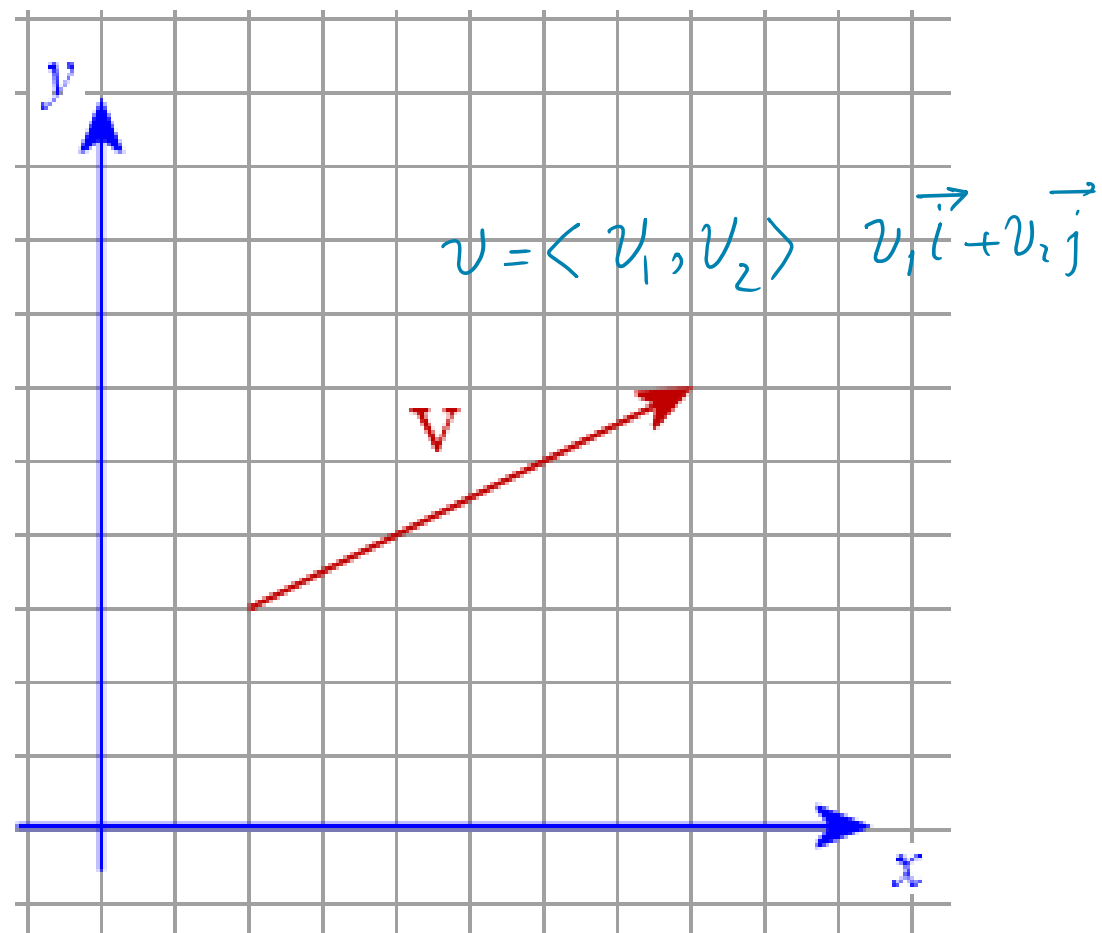
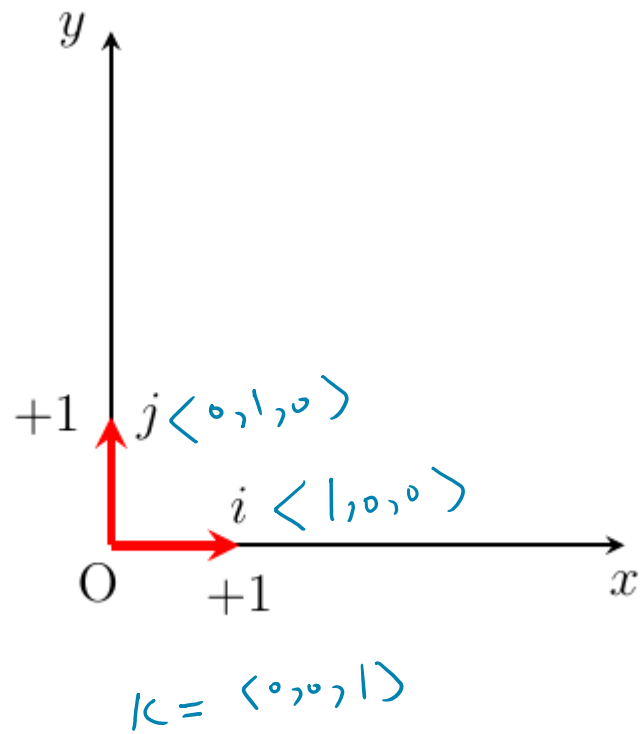


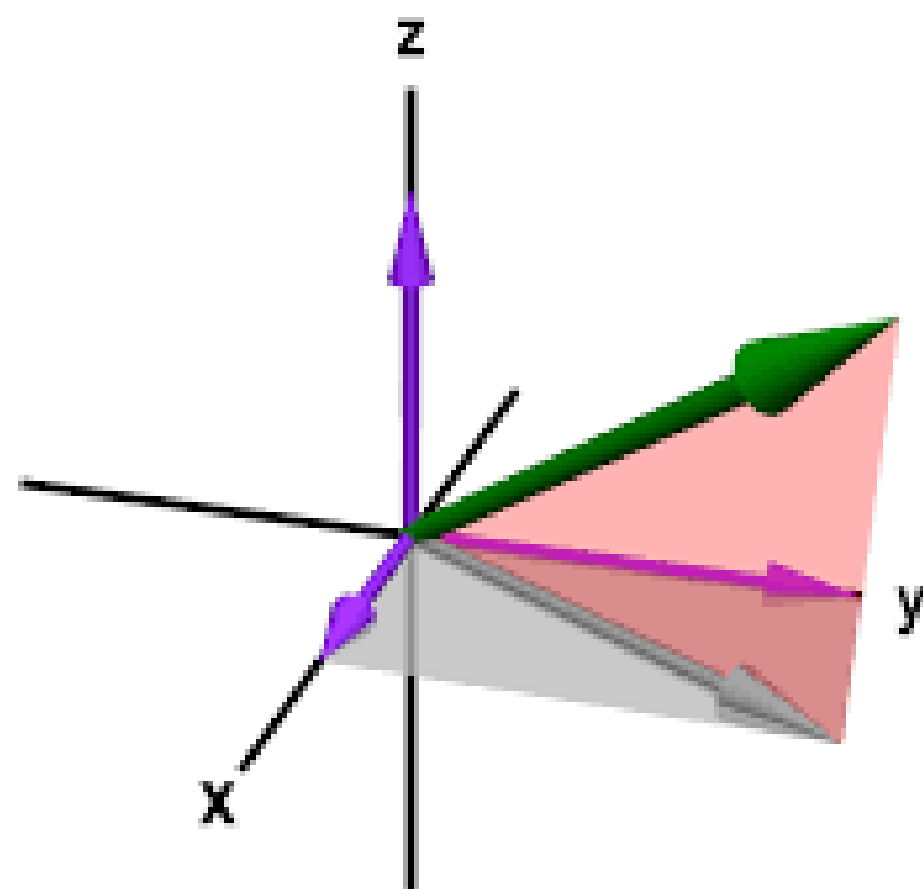
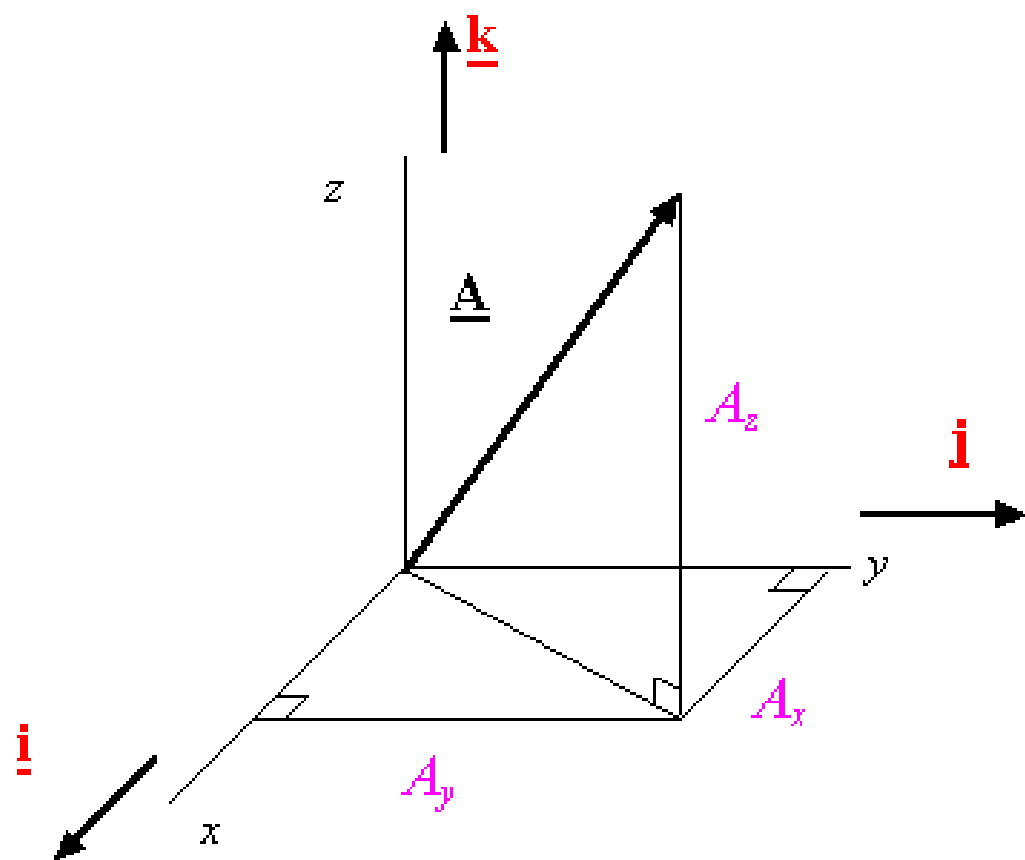
$$L = \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \times \frac{dx}{dt} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{cases} x = r(1 - \sin \theta) \Rightarrow \frac{dx}{d\theta} \\ y = r(1 - \cos \theta) \Rightarrow \frac{dy}{d\theta} \end{cases}$$

$\Rightarrow$ ok







مثال: بردار واحد هم جهت با بردار $\langle -2, 6 \rangle$ را پیدا کنید.

$$\vec{v} = 6\vec{i} - 2\vec{j}$$

$$\left. \begin{array}{l} v = \langle v_1, v_2, v_3 \rangle \\ |v| = \sqrt{v_1^2 + v_2^2 + v_3^2} \end{array} \right\} \frac{v}{|v|} \Rightarrow \text{بردار واحد}$$

$$\left. \begin{array}{l} \langle \frac{v_1}{|v|}, \frac{v_2}{|v|} \rangle = \\ |v| = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10} \end{array} \right\} \langle \frac{6}{2\sqrt{10}}, \frac{-2}{2\sqrt{10}} \rangle$$

$$a = \langle a_1, a_2 \rangle$$

$$b = \langle b_1, b_2 \rangle$$

$$a+b = \langle a_1+b_1, a_2+b_2 \rangle$$

$$a+(b+c) = (a+b)+c$$

$$a+0 = a$$

$$a+(-a) = 0$$

$$c(a+b) = ca+cb$$

$$(cd)a = c(da)$$

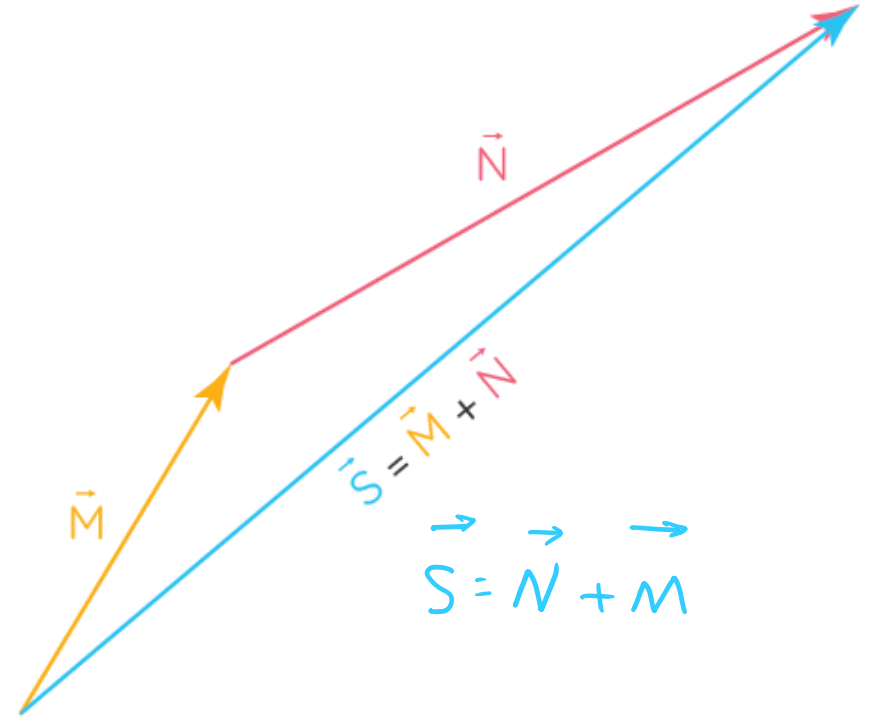
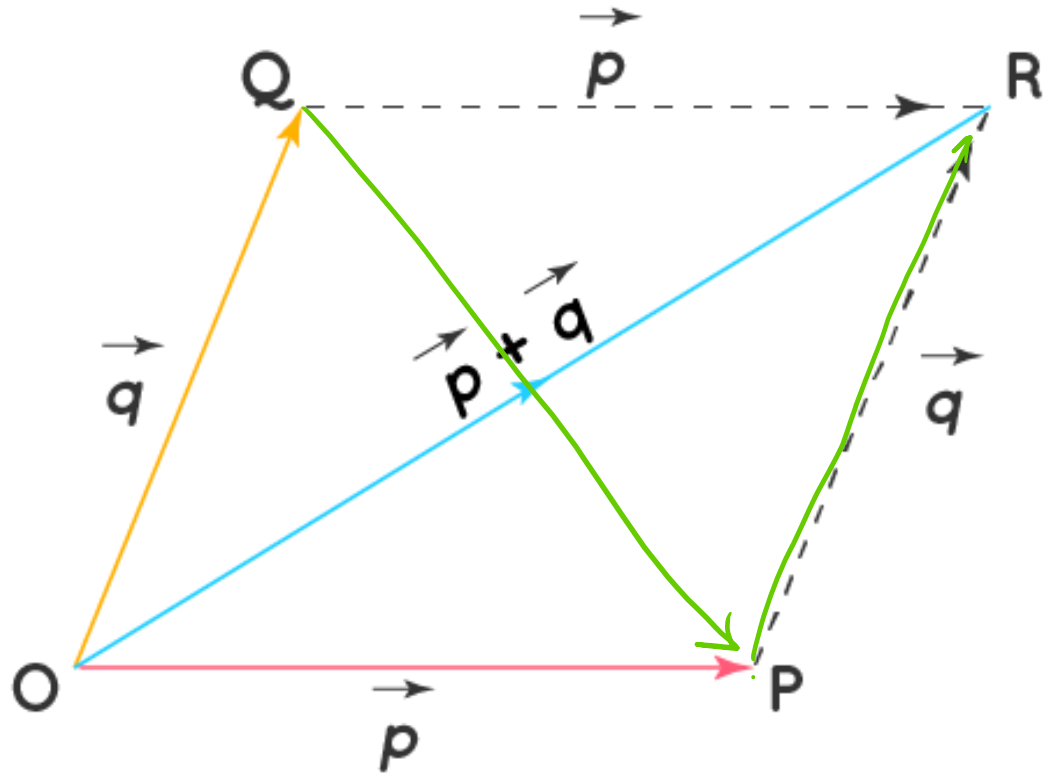
$$1a = a$$

$$a+b = b+a \quad \text{خاصیت جابجایی دارد}$$

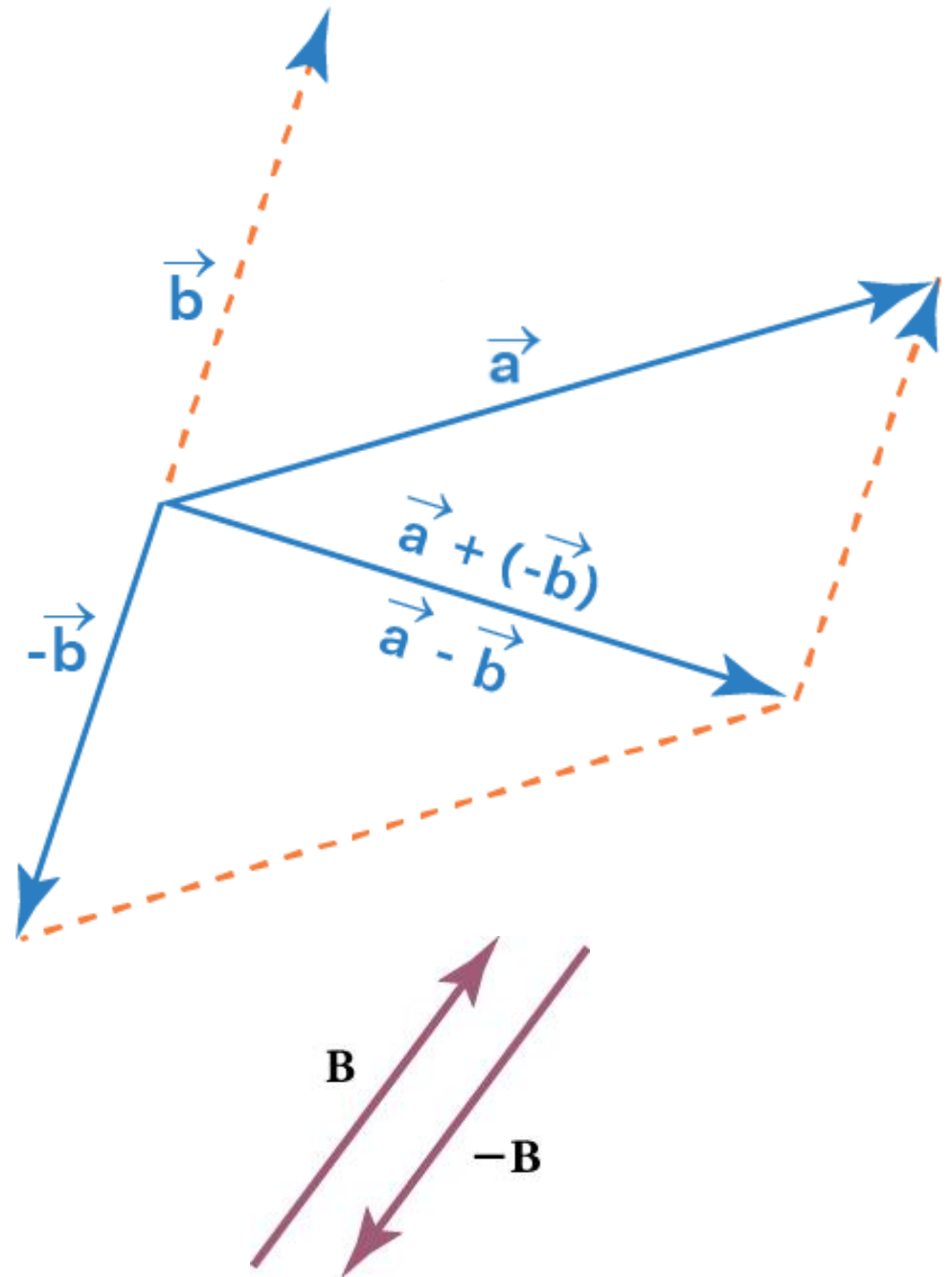
جهت
بردار

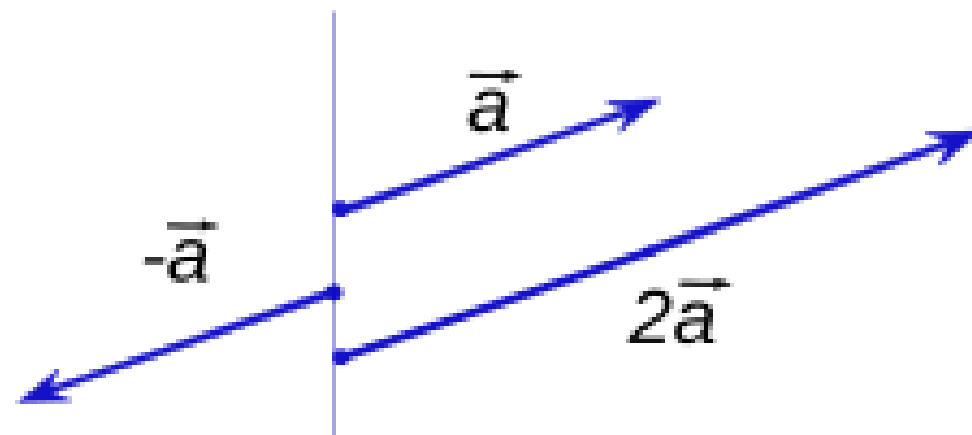
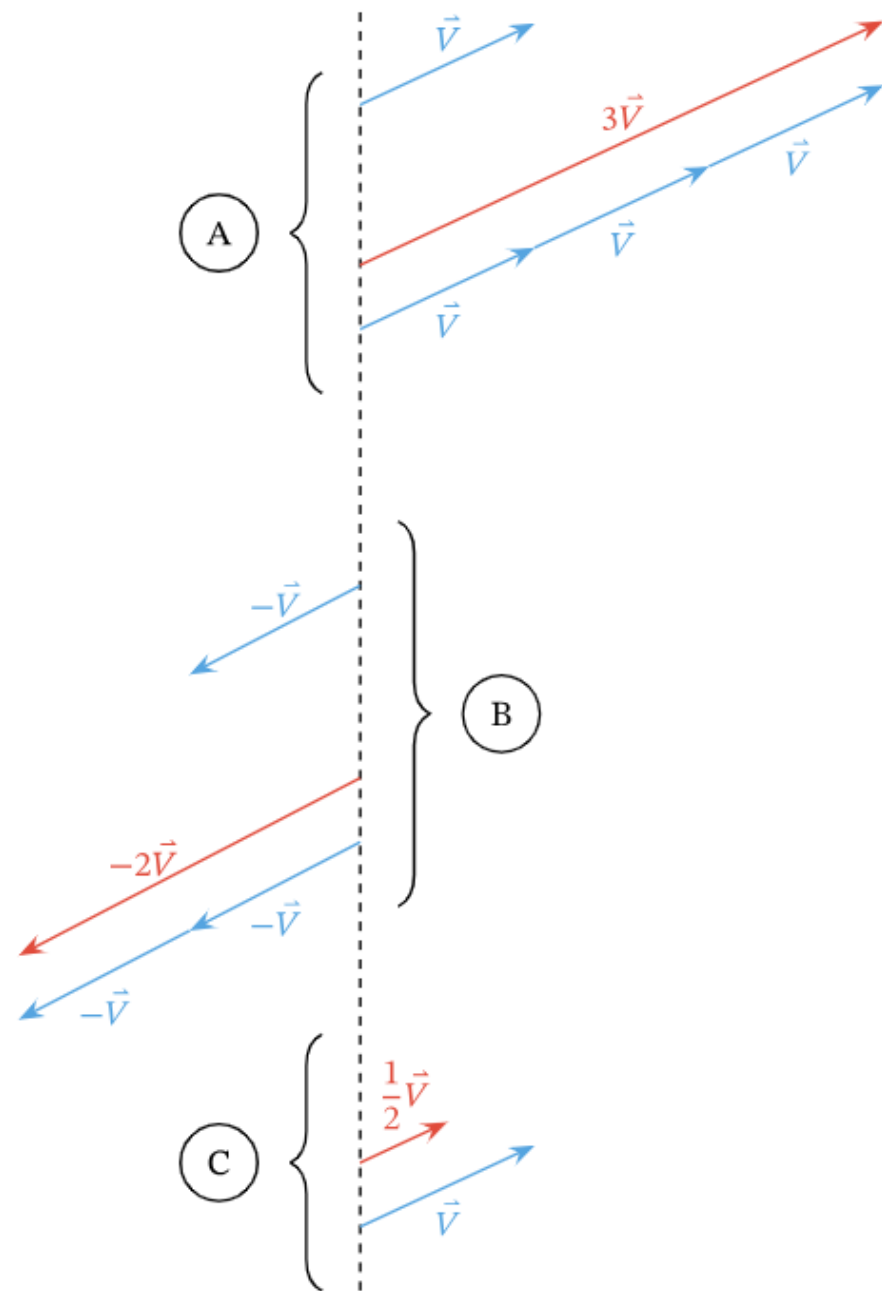
$$\Rightarrow \vec{p} + \vec{q} = \vec{q} + \vec{p}$$

خاصیت جابجایی



$$a - b = a + (-b)$$





Dot product ضرب نقطه‌ای

$$a = \langle a_1, a_2, \dots, a_n \rangle$$

$$b = \langle b_1, b_2, \dots, b_n \rangle$$

$$a \cdot b = \underbrace{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}_{\text{مجموع}}$$

$$a \cdot a = |a|^2$$

$$a \cdot b = b \cdot a \quad \text{جابجایی}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$\left. \begin{array}{l} a = \langle 2, -1, 2 \rangle \\ b = \langle 1, -1, 0 \rangle \end{array} \right\} \text{زاویه بین دو بردار} = ?$$

$$|a| = \sqrt{9} = 3$$

$$|b| = \sqrt{2}$$

$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{2 + 1 + 0}{\sqrt{2} \times 3} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4} \text{ rad } 45^\circ$$

$$a: 2\vec{i} + 2\vec{j} - \vec{k}$$

$$b: 5\vec{i} - 4\vec{j} + 2\vec{k}$$

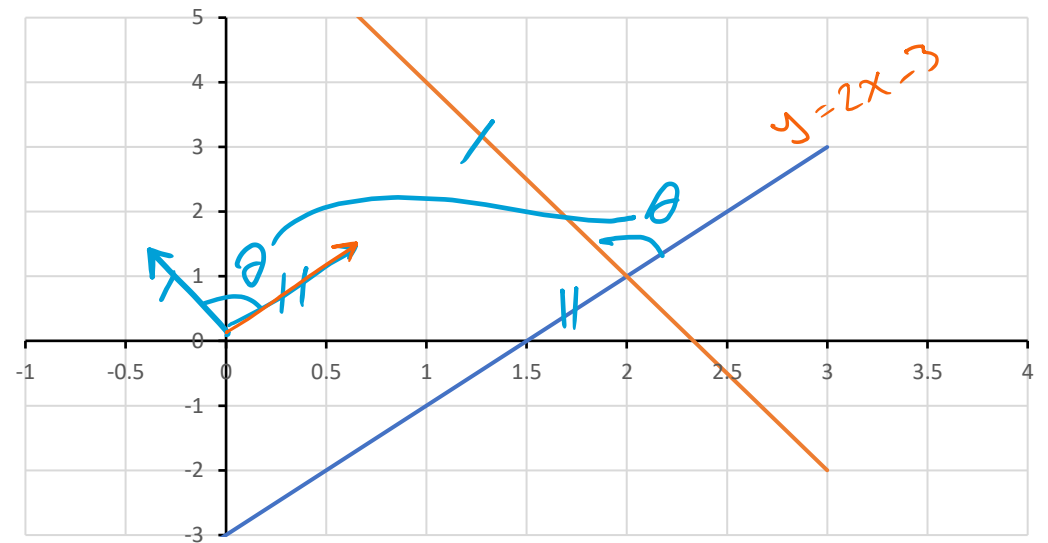
برهم عمودند

$$a \cdot b = 0 \Rightarrow 10 - 8 - 2 = 0$$

$$\begin{cases} 2x - y = 3 \\ y = 2x - 3 \end{cases} \quad a: \langle 1, 2 \rangle$$

$$\begin{cases} 3x + y = 7 \\ y = 7 - 3x \end{cases} \quad b: \langle 1, -3 \rangle$$

$$y = 7 - 3x$$

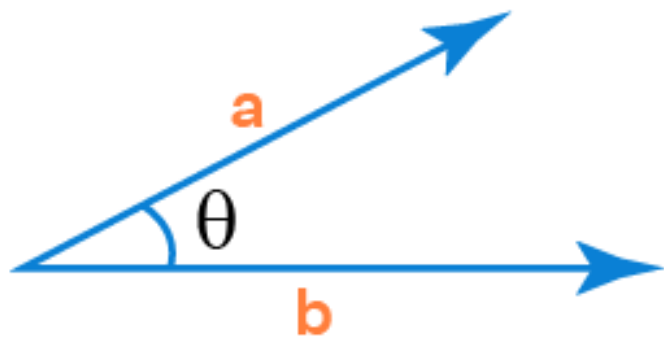


$$\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{1 - 6}{\sqrt{5} \times \sqrt{10}} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow$$

$$\theta = 45^\circ$$

$$\theta = 135^\circ$$

ویژگی‌های بردارها



$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

(OR)

$$\sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\text{ماتریس همانی} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{ماتریس مقدر} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ 1 & 5 \end{bmatrix}$$

جمع و تفریق دو ماتریس
درایه به درایه

ضرب عدد در ماتریس

$$2 \times \begin{bmatrix} 3 & 2 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & -18 \end{bmatrix}$$

$$\underbrace{A}_{n \times k} \times \underbrace{B}_{k \times m} = \underbrace{C}_{n \times m}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1 \times 10 + 2 \times 20 + 3 \times 30 & 1 \times 11 + 2 \times 21 + 3 \times 31 \\ 4 \times 10 + 5 \times 20 + 6 \times 30 & 4 \times 11 + 5 \times 21 + 6 \times 31 \end{bmatrix}_{2 \times 2}$$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \\ 6 & 8 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 4 & 8 \\ 5 & 6 & 7 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(A \times B)^T = B^T \times A^T \neq A^T \times B^T$$

$$A = A^T \quad * \text{ماتريس متقارن}$$

$$\text{مجموع عناصر قطر اصلی} \leftarrow \text{tr}(A) = \sum_{i=1}^n a_{ii} \quad * \text{اثر ماتريس}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{tr}(A) = 1 + 5 + 9 = 15$$

$$\|A\|_F = \sqrt{\sum a_{ij}^2} = \sqrt{\text{Tr}(AA^T)}$$

* نرم ماتریس

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = |A| = ad - bc$$

* دترمینان ماتریس

$$\det \begin{bmatrix} 8 & 3 \\ 4 & 2 \end{bmatrix} = \frac{8 \times 2}{16} - \frac{3 \times 4}{12} = 4$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$\text{sgn} = (-1)^{i+j}$$

$$A = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

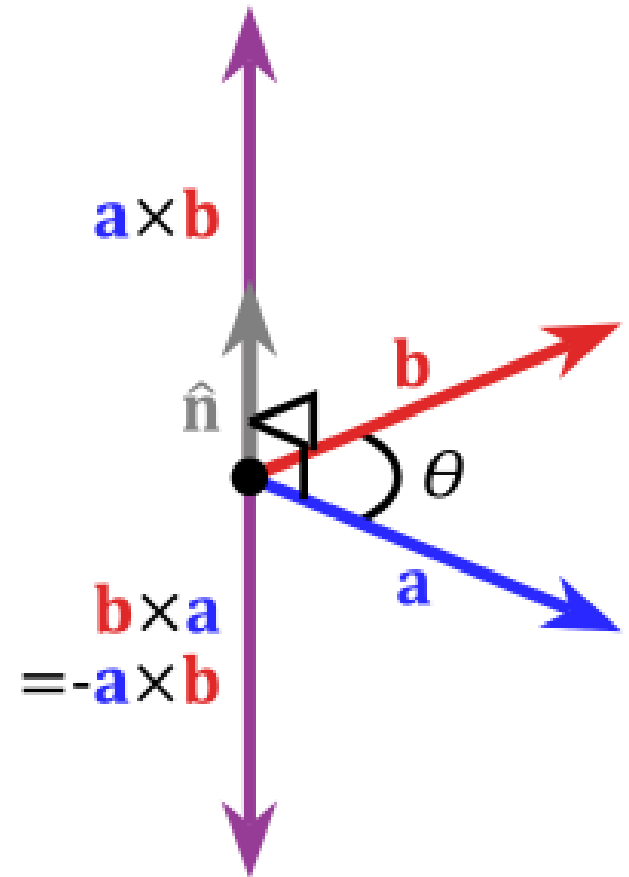
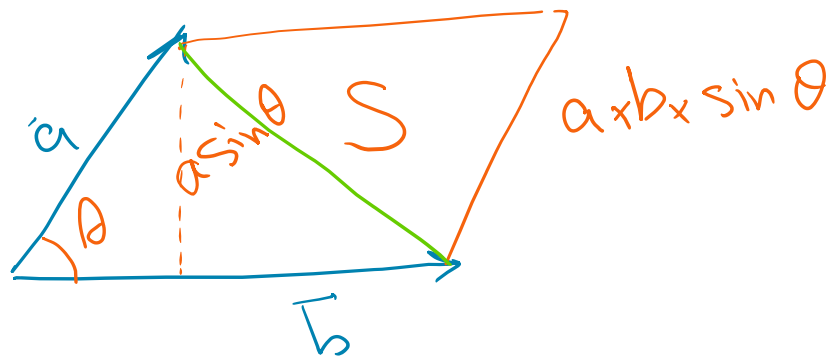
$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

$$a = \langle a_1, a_2, a_3 \rangle \quad a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$b = \langle b_1, b_2, b_3 \rangle \quad b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$|a \times b| = |a||b| \sin \theta$$



$$A(0,0,0)$$

$$B(1,0,2) \Rightarrow \text{مساحت مثلث؟}$$

$$C(2,-2,0)$$

$$A \rightarrow A^{-1}$$

$$A \cdot A^{-1} = I$$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1} \times A^{-1}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{8-9} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_{3 \times 3}^{-1} = \frac{1}{|A|} [Adjoint]$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^T =$$

$$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \quad (-1)^{i+j}$$

$$adj(A) = \begin{bmatrix} + | e h | - | b h | + | b e | \\ + | f i | - | c i | + | c f | \\ - | d g | + | a g | - | a d | \\ - | f i | + | c i | - | c f | \\ + | d g | - | a g | + | a d | \\ + | e h | - | b h | + | b e | \end{bmatrix}$$

ماتریس وارون

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

وارون ماتریس قطری

$$A \underline{x} = \lambda \underline{x} \quad \begin{array}{l} \text{مقدار ویژه} \downarrow \\ \text{بیدار ویژه} \end{array}$$

$$A \underline{x} - \lambda \underline{x} = \underline{0}$$

$$(A - \lambda I) \underline{x} = \underline{0} \quad \leftarrow$$

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

بیدار ویژه

مقدار ویژه

مقادیر ویژه و بیدار ویژه

$$a = (1, 1)$$

$$a = (a_1, a_2)$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \Rightarrow \begin{array}{c} A \\ \hline \end{array} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{array}{c} I \\ \hline \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

$$(1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda = \begin{cases} -2 \\ 5 \end{cases} \Rightarrow x_1 = x_2 \Rightarrow \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-(-2)}{3} & 4 \\ 3 & \frac{2-(-2)}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} 3x_1 + 4x_2 = 0 \Rightarrow x_1 = -\frac{4}{3}x_2 \\ 3x_1 + 4x_2 = 0 \end{array}$$

$$\begin{cases} -4x_1 + 4x_2 = 0 \Rightarrow x_1 = x_2 \\ 3x_1 - 3x_2 = 0 \Rightarrow x_1 = x_2 \end{cases}$$

$$\begin{bmatrix} -\frac{4}{3}a \\ a \end{bmatrix}$$

دستگاه معادلات خطی و ماتریس

$$A \times X = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\underline{A} X = b$$

$$\underbrace{A^{-1} \times A}_{I} \times X = A^{-1}b \Rightarrow X = A^{-1}b$$

$$\begin{cases} x_1 - x_2 = -1 \\ 3x_1 + x_2 = 9 \end{cases} \Rightarrow x_1 = -1 + x_2 \xrightarrow[\text{در معادله}]{\text{قادر باین}} 3(-1 + x_2) + x_2 = -1 \Rightarrow \begin{cases} x_2 = 3 \\ x_1 = 2 \end{cases}$$

$$AX = b$$

$$\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} [\text{adj}(A)] = \frac{1}{1 - (-3)} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

روش کرامر

$$x_1 = \frac{\begin{vmatrix} -1 & -1 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{-1 - (-9)}{1 - (-3)} = \frac{8}{4} = 2$$

$$x_2 = \frac{\begin{vmatrix} 1 & -1 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{9 - (-3)}{4} = \frac{12}{4} = 3$$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & \textcircled{-1} & -1 \\ \textcircled{3} & 1 & 9 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} \textcircled{4} & 0 & 8 \\ 3 & 1 & 9 \end{array} \right] \xrightarrow{\frac{R_1}{4}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ \textcircled{3} & 1 & 9 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & \textcircled{2} \\ 0 & 1 & \textcircled{3} \end{array} \right] \begin{matrix} x_1 \\ x_2 \end{matrix}$$

I باء

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 5 \\ 3x_1 + 5x_2 + 6x_3 = 7 \\ 2x_1 + 4x_2 + 3x_3 = 8 \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}} = -15 \quad x_2 = 8 \quad x_3 = 2$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix} = 1(5 \times 3 - 6 \times 4) - 3(3 \times 3 - 6 \times 2) + (-2)(3 \times 4 - 5 \times 2)$$

برای هر چه مقدار از a دستگاه زیر بیشترین جواب دارد

$$\begin{cases} x - y + az = 0 \\ 3x - y + 2z = 0 \\ 6x - 4y + 5z = 0 \end{cases}$$

$$AX = 0$$

$$|A| = 0 \Rightarrow \text{بیشترین جواب}$$

(نقطه شروع) P_0
 $\vec{v} \langle a, b, c \rangle$

$$y - y_0 = m(x - x_0)$$

معادلات خط و صفحه در مقایسه به بعد

$$\vec{r} = \vec{r}_0 + \vec{a}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

معادله بردار خط

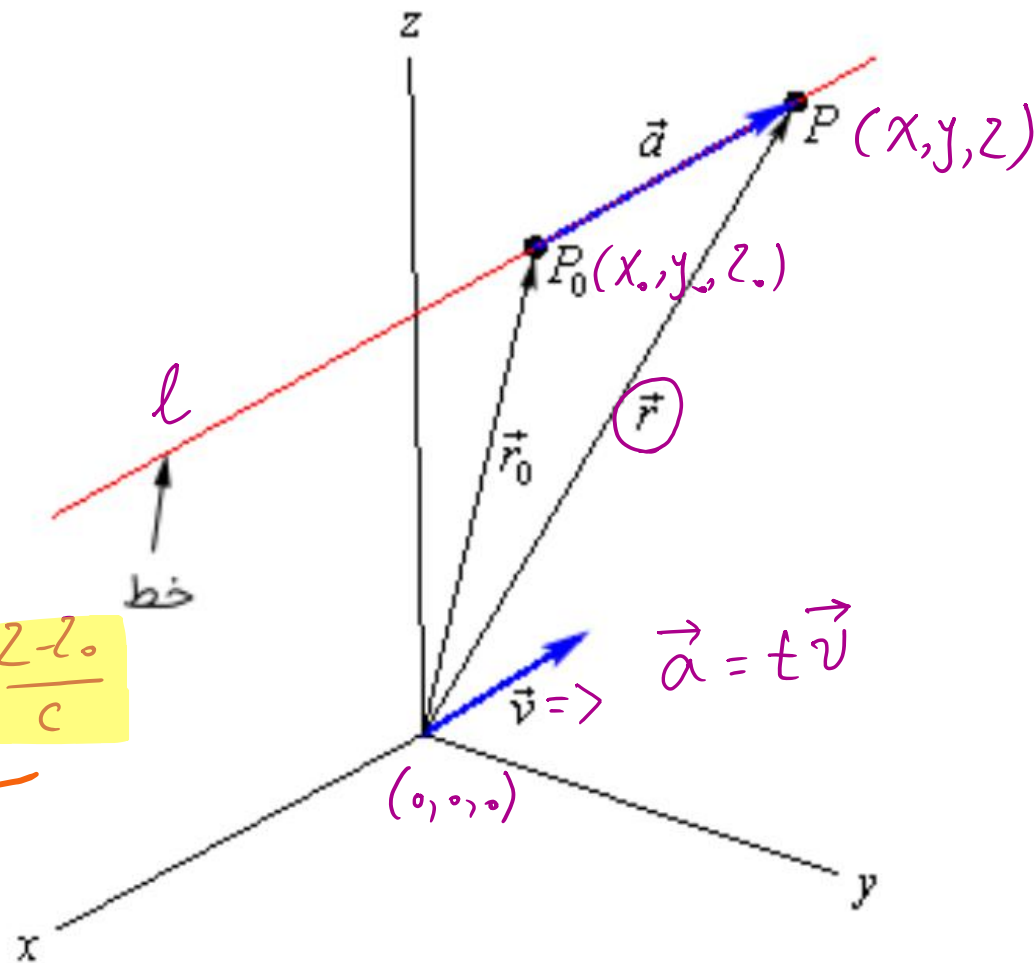
$$(x_0 + ta)\vec{i} + (y_0 + tb)\vec{j} + (z_0 + tc)\vec{k}$$

معادله پارامتر خط

$$\left\{ \begin{array}{l} x = x_0 + ta \Rightarrow t = \frac{x - x_0}{a} \\ y = y_0 + tb \Rightarrow t = \frac{y - y_0}{b} \\ z = z_0 + tc \Rightarrow t = \frac{z - z_0}{c} \end{array} \right.$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

معادلات متقارن خط



مثال) $P_0(2, 2.4, 3.5)$

$$\vec{a} = \underbrace{3\vec{i}}_a + \underbrace{2\vec{j}}_b - \underbrace{1\vec{k}}_c$$

$$r = r_0 + t v$$

$$\langle x, y, z \rangle = \langle x_0 + t a, y_0 + t b, z_0 + t c \rangle$$

$$\langle 2 + 3t, 2.4 + 2t, 3.5 + (-1)t \rangle$$

معادله بردار $(2+3t)\vec{i} + (2.4+2t)\vec{j} + (3.5-t)\vec{k}$

$$\vec{a} \begin{cases} A(-4, -6, 1) \\ B(-2, 0, -3) \end{cases} \rightarrow \vec{b} \begin{cases} C(10, 18, 4) \\ D(5, 3, 14) \end{cases} \Rightarrow$$

$$\vec{a} = \langle 2, 6, -4 \rangle$$

$$\vec{b} = \langle -5, -15, 10 \rangle$$

$$\frac{b_1}{a_1} = \frac{-5}{2}$$

$$\frac{b_2}{a_2} = \frac{-15}{6} = \frac{-5}{2}$$

$$\frac{b_3}{a_3} = \frac{10}{-4} = \frac{-5}{2}$$

این دو بردار موازی هستند

$$\vec{b} = \left(\frac{-5}{2} \right) \vec{a}$$

$$\left. \begin{array}{l} \ell_1: x=3+2t \quad y=4-t \quad z=1+3t \\ \ell_2: x=1+4s \quad y=3-2s \quad z=4+5s \end{array} \right\} \Rightarrow \begin{array}{l} v_1 = \langle 2, -1, 3 \rangle \\ v_2 = \langle 4, -2, 5 \rangle \end{array}$$

$v_1 \nparallel v_2$
موازی نیستند

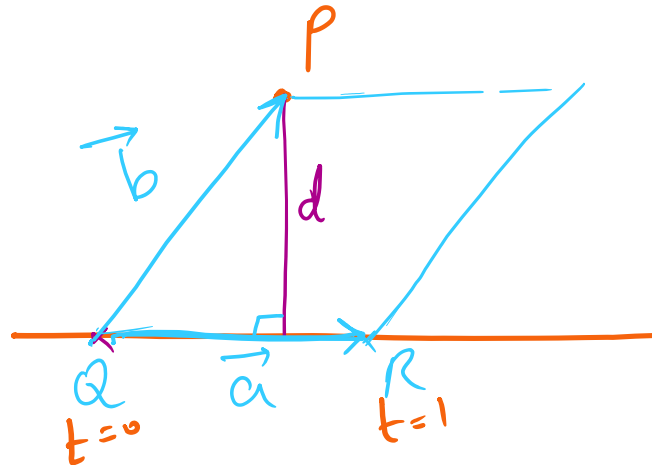
$$P(4, 1, -2)$$

فاصله نقطه P از خط ℓ_1

$$x=1+t$$

$$y=3-2t$$

$$z=4-3t$$



$$|a \times b| = d \times |a| \Rightarrow d = \frac{|a \times b|}{|a|}$$

$$t=0 \Rightarrow Q = \langle 1, 3, 4 \rangle$$

$$t=1 \Rightarrow R = \langle 2, 1, 1 \rangle$$

$$\vec{b} = \langle 3, -2, -6 \rangle$$

$$\vec{a} = \langle 1, -2, -3 \rangle$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix} = \underbrace{(12-6)}_6 \vec{i} - \underbrace{(-6+9)}_{-3} \vec{j} + \underbrace{(-2+6)}_4 \vec{k}$$

$$* \Rightarrow d = \frac{\sqrt{36+9+16}}{\sqrt{1+4+9}} = \frac{\sqrt{61}}{\sqrt{14}}$$

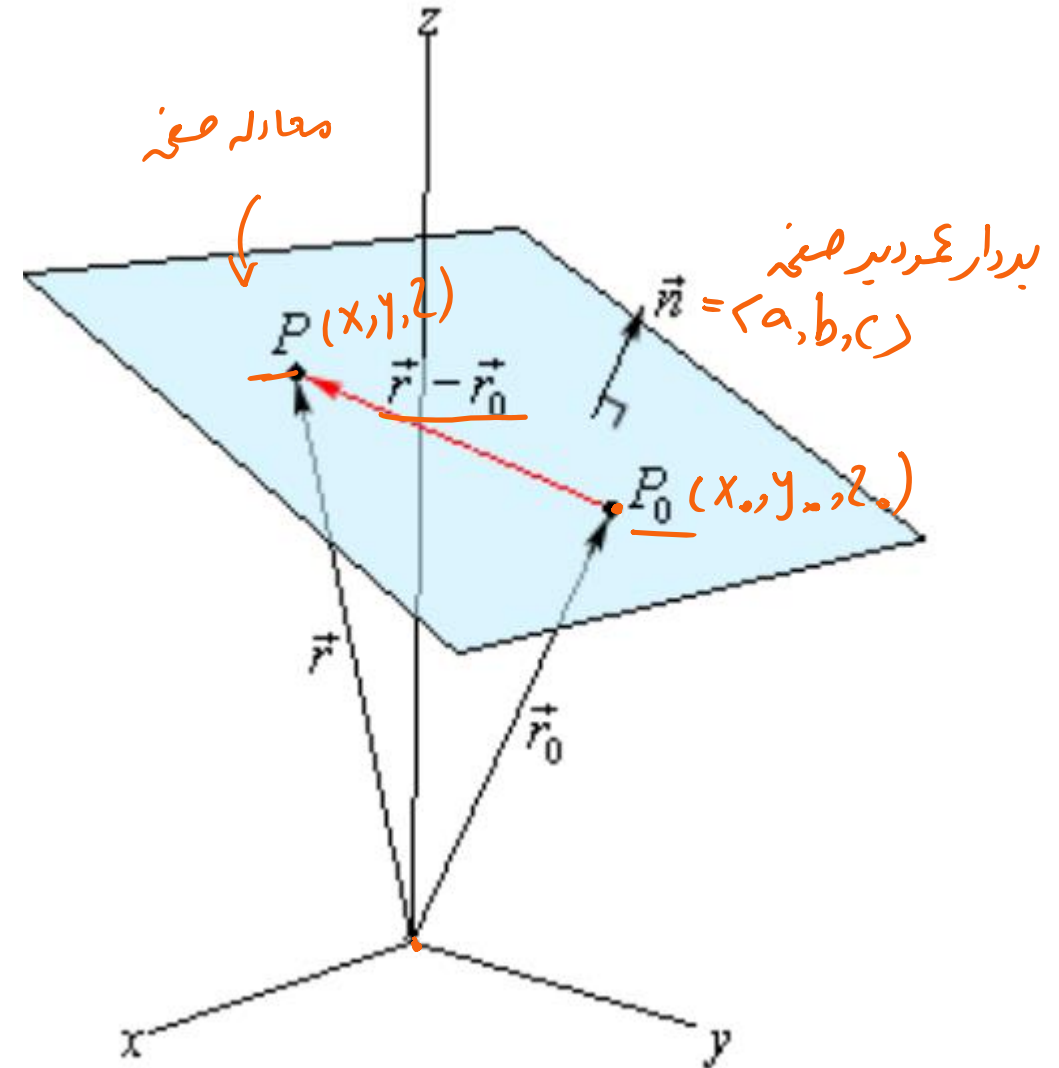
مستوی دایره‌ای به مرکز r_0 و بردار عمود بر صفحه n است $\Rightarrow n \perp (r - r_0)$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

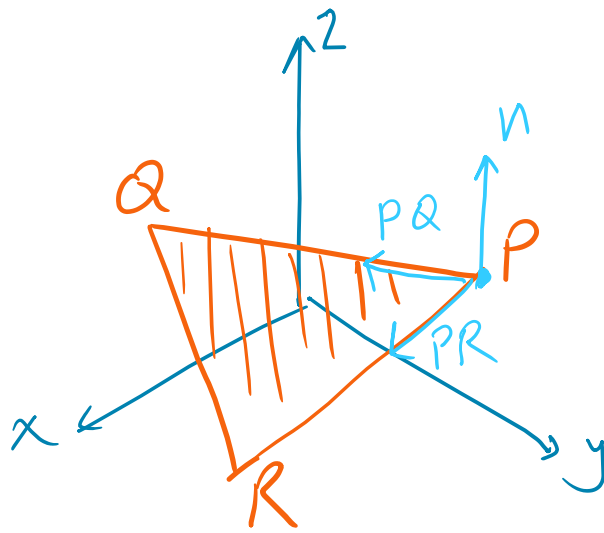
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz - \underbrace{ax_0 + by_0 + cz_0}_{d} = 0$$

$$\boxed{ax + by + cz - d = 0}$$



$$\begin{aligned} R & (5, 2, 0) \\ Q & (3, -1, 6) \\ P & (1, 3, 2) \end{aligned}$$



$$\vec{PQ} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

$$n = PQ \times PR = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \underbrace{12}_{a} \vec{i} + \underbrace{20}_{b} \vec{j} + \underbrace{14}_{c} \vec{k}$$

$$ax + by + cz - d = 0$$

$$12x + 20y + 14z - (12 \times 1 + 20 \times 3 + 14 \times 2) = 0$$

$$6x + 10y + 7z - 50 = 0$$

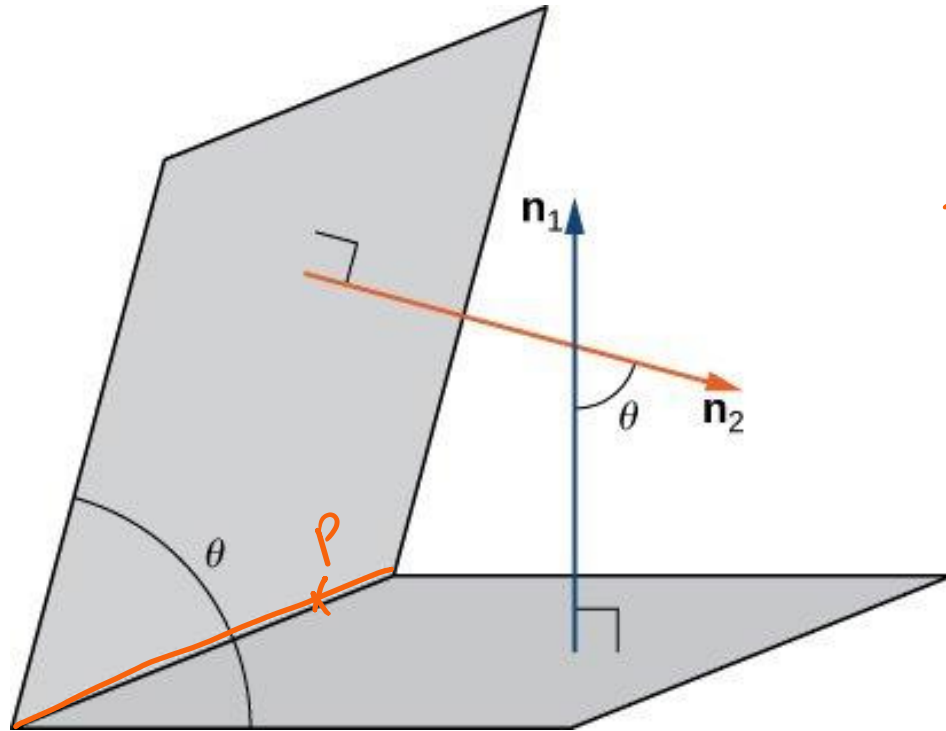
* تمرین

نقطه اشتراک خط زیر با صفحه $4x + 5y - 2z = 18$

$$\begin{cases} x = 2 + 3t \\ y = -4t \\ z = 5 + t \end{cases}$$

* معادله خط گذرنده از نقطه $(6, 0, 1)$ و عمود بر صفحه $x + 3y + 2z = 5$ را مشخص کنید

دو صفحه موازی و عمود و زاویه بین دو صفحه



1) $n_1 = 1 < n_2$ موازی
2) $n_1 \cdot n_2 = 0$ عمود

3) $n_1 \cdot n_2 = |n_1| \cdot |n_2| \times \cos \theta$

$\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|}$

مثال
$$\begin{cases} x + 4y - 3z = 1 \\ -3x + 6y + 7z = 0 \end{cases}$$

$n_1 = (1, 4, -3)$ $n_2 = (-3, 6, 7)$

تمرین *

زاویه بین دو صفحه

مقادیر خط
استعداد

$$\begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases}$$

$\theta = \cos^{-1} \left(\frac{2}{\sqrt{42}} \right) \approx 72^\circ$

$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$

$n \cdot n_2 = 0$

فاصله نقطه از صفحه

برای اسکالر
 n و v

$$d = \frac{|n \cdot v|}{|n|}$$

$n = \langle a, b, c \rangle$

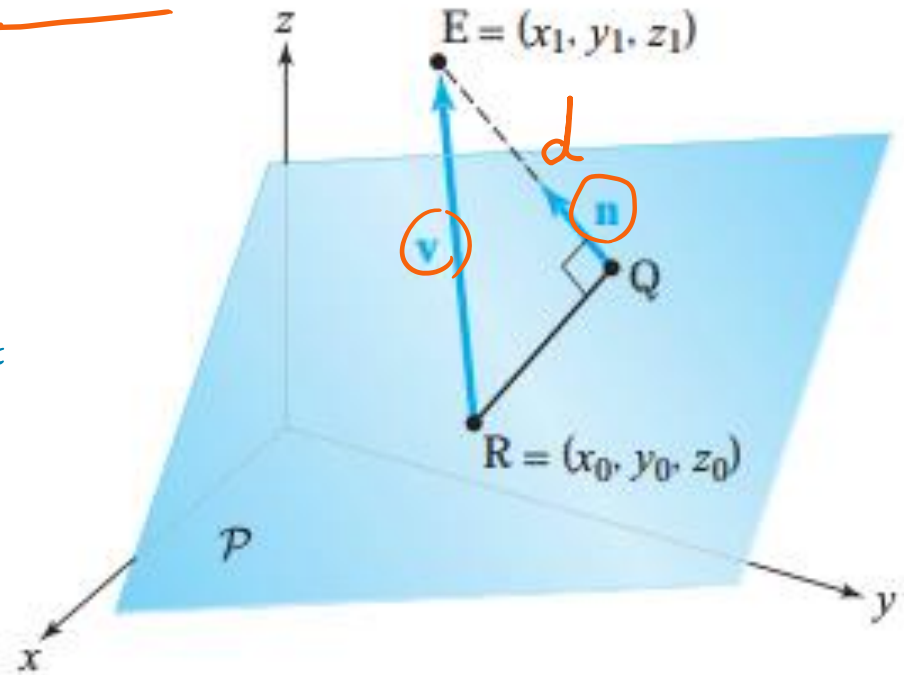
$$d = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$v = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

* مثال: فاصله نقطه $(1, -2, 4)$ از صفحه $3x + 2y + 6z = 5$

$n = \langle 3, 2, 6 \rangle$

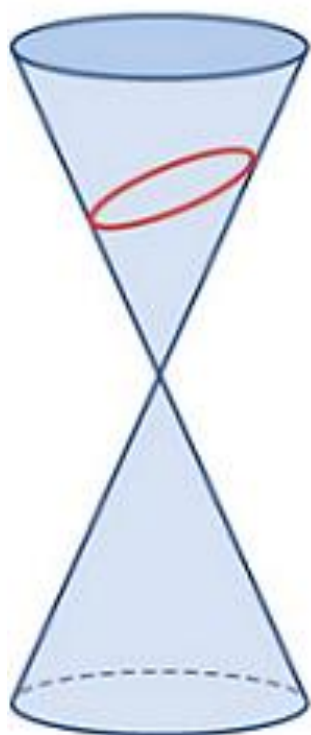
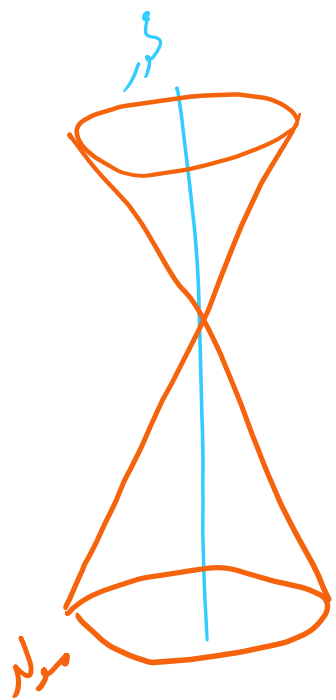
$$d = \frac{|3 \times 1 + 2(-2) + 6 \times 4 - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{18}{7}$$



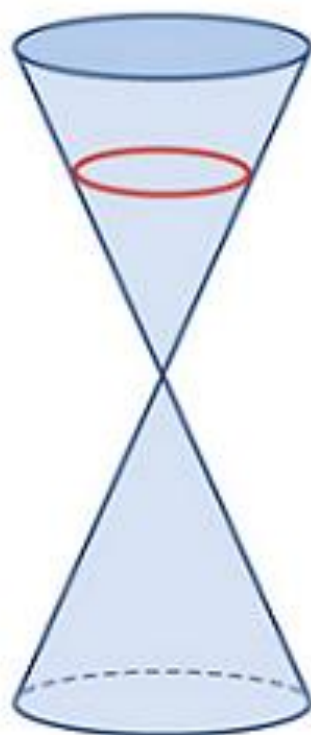
$$\begin{cases} \underline{a}x + \underline{b}y + cz + d_1 = 0 \\ \underline{a}x + \underline{b}y + cz + d_2 = 0 \end{cases} \Rightarrow d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\begin{cases} 10x + 2y - 2z = 5 \\ 5x + y - z = 1 \end{cases} \Rightarrow \text{فاصله این دو صفحه را بیابید}$$

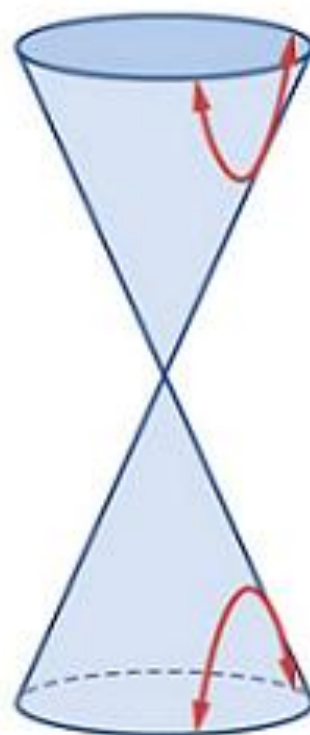
* تمرین



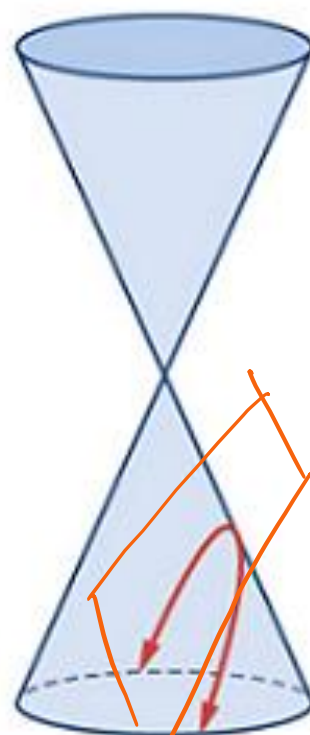
بیضی



دایره

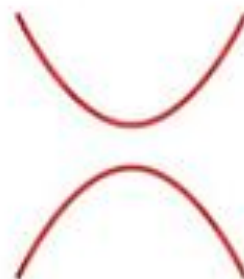
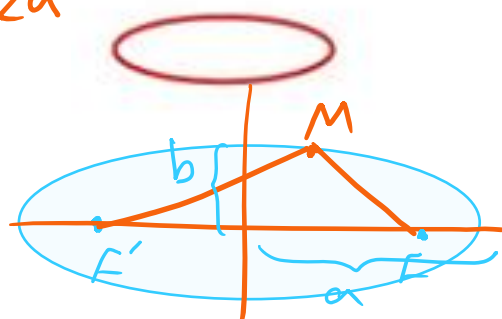


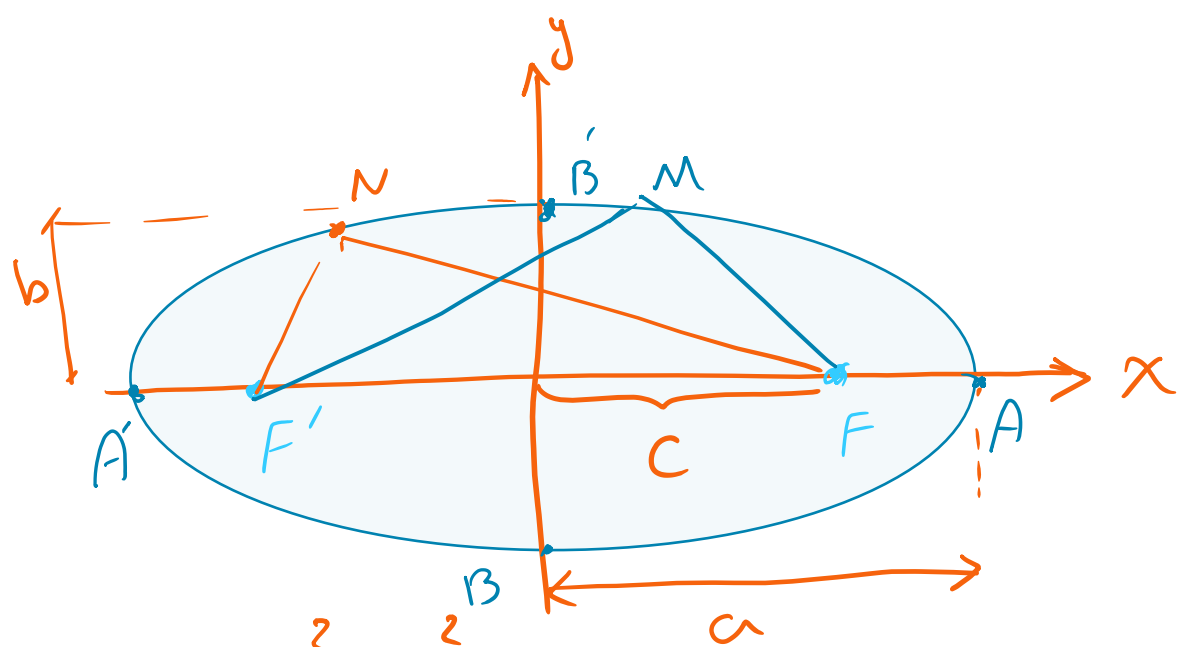
هذلولی



سهمی

$$|MF'| + |MF| = 2a$$





معادله بیضی

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$|MF'| + |MF| = 2a$$

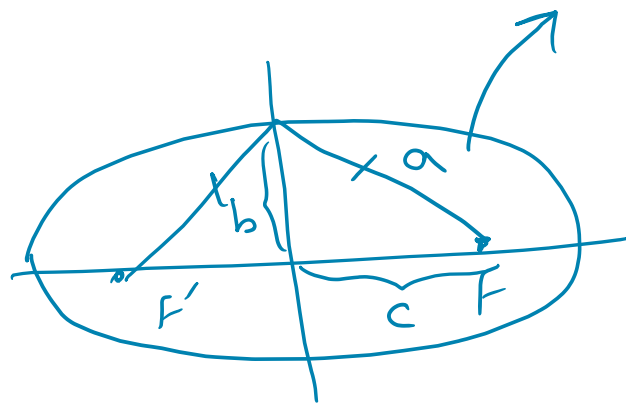
$$|NF'| + |NF| = 2a$$

$$|A'A| = 2a$$

$$|B'B| = 2b$$

$$|F'F| = 2c$$

$$a^2 = b^2 + c^2$$



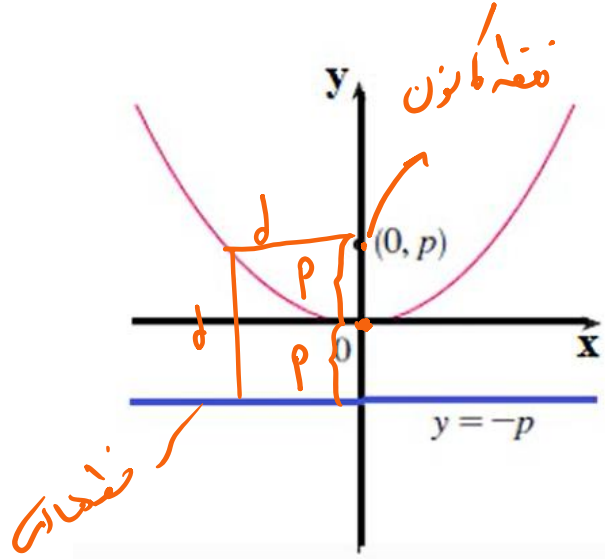
* تمرین

فاصله کانونی؟

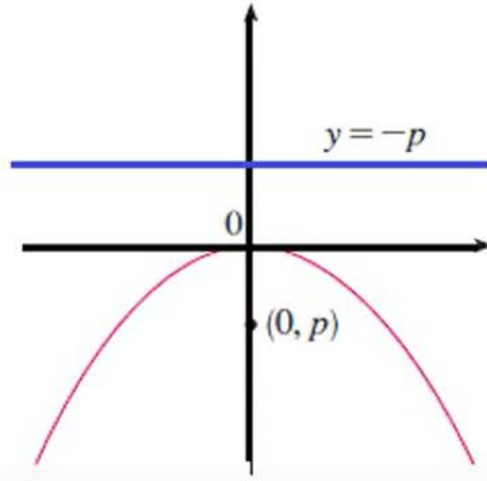
$$9x^2 + 16y^2 = 144$$

پارسه (Parabola)

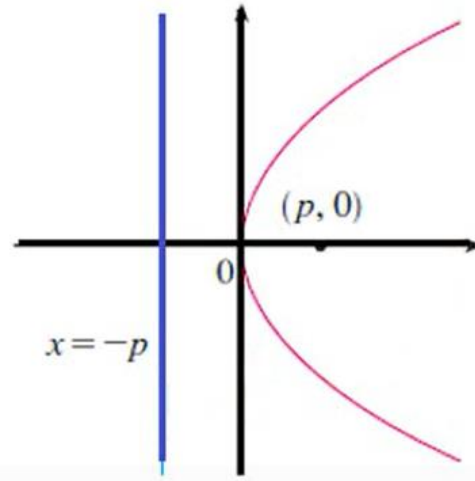
مکان هندسه نقاط که نامله آئنا از يك نفعه ثابت (كارزون) و يك خط ثابت (خط هاله) سكله برابر اس.



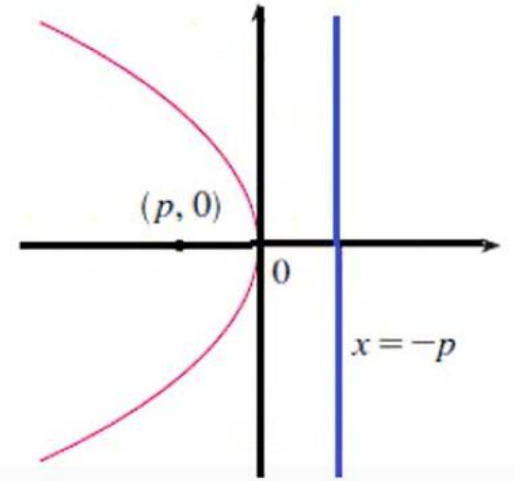
$$x^2 = 4py$$



$$x^2 = -4py$$



$$y^2 = 4px$$



$$y^2 = -4px$$

$$(y - \beta)^2 = 4p(x - \alpha)$$

(α, β) مختصات مركز

انواع ریب (سطح)

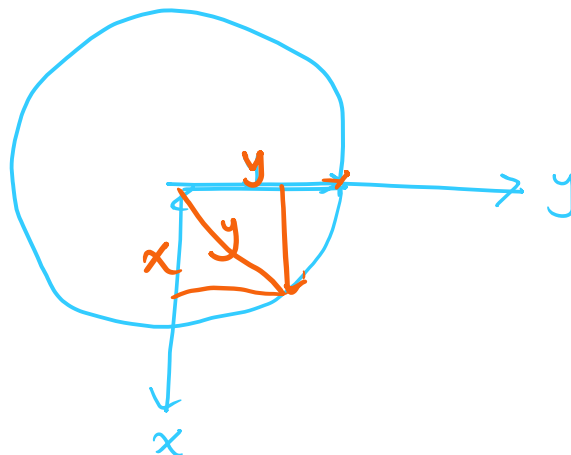
۱- سطوح حاصل از دوران

مثال: $Z = y^2$ این منحنی را حول محور Z دوران می دهیم = معادله سطح ؟
در صفحه yz

$$\left\{ \begin{array}{l} \sqrt{Z} \\ \sqrt{x^2 + y^2} \end{array} \right\} \text{ شعاع دوران}$$

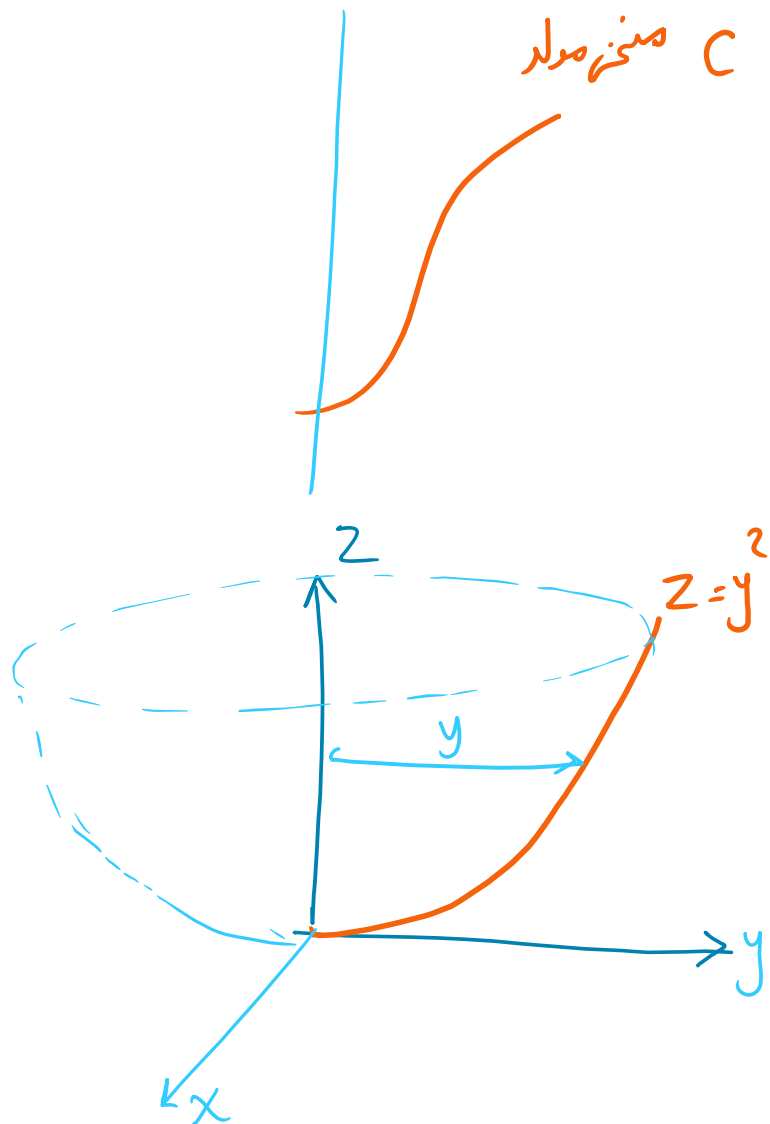
$$\sqrt{x^2 + y^2} = \sqrt{Z}$$

$$\boxed{x^2 + y^2 = Z}$$



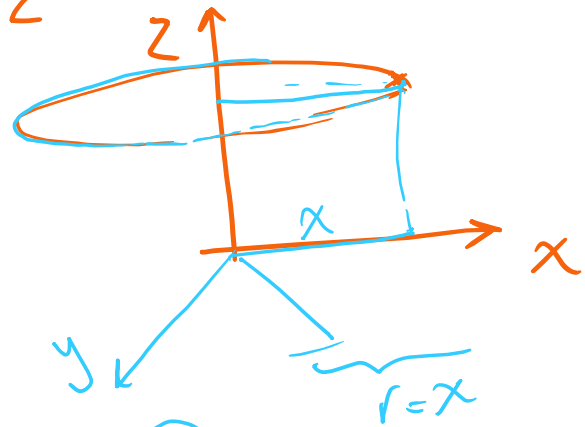
l محور دوران

C منحنی مولد



مثال :

$$\left. \begin{array}{l} \sqrt{x^2 + y^2} \\ xz \downarrow z \end{array} \right\} \Rightarrow (\sqrt{x^2 + y^2}) \cdot z^2 = 1 + \sin(x^2 + y^2)$$



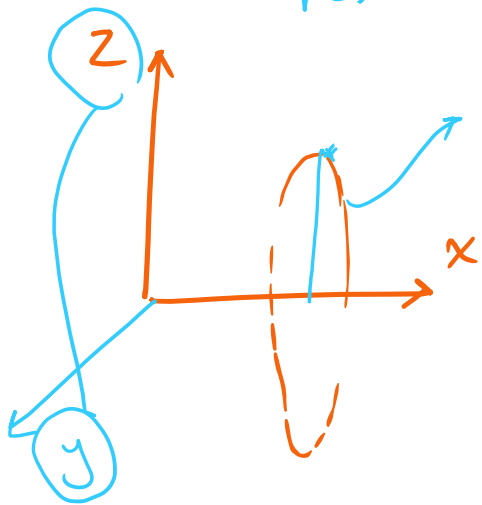
$$\begin{array}{l} z = z \\ r = \sqrt{x^2 + y^2} \xrightarrow[y \rightarrow \infty]{zx} \boxed{r = x} \\ r = x \end{array}$$

$$\cancel{x} z^2 = 1 + \sin \cancel{x}^2$$

xz

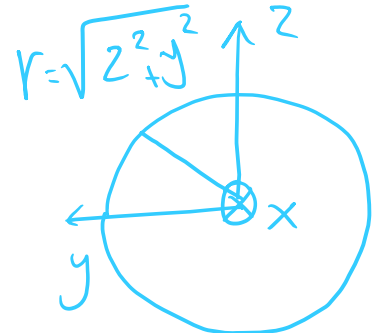
الف - دوران حول محور z

ب - دوران حول محور x

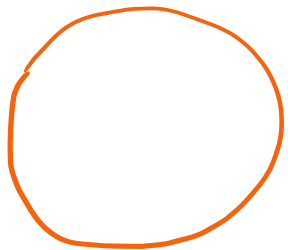


$$\Rightarrow \begin{cases} x = x \\ r = \sqrt{z^2 + y^2} \end{cases} \xrightarrow[y \rightarrow \infty]{zx} r = 2$$

$$\left. \begin{array}{l} x \rightarrow x \\ z \rightarrow \sqrt{z^2 + y^2} \end{array} \right\} \Rightarrow x(z^2 + y^2) = 1 + \sin x^2$$



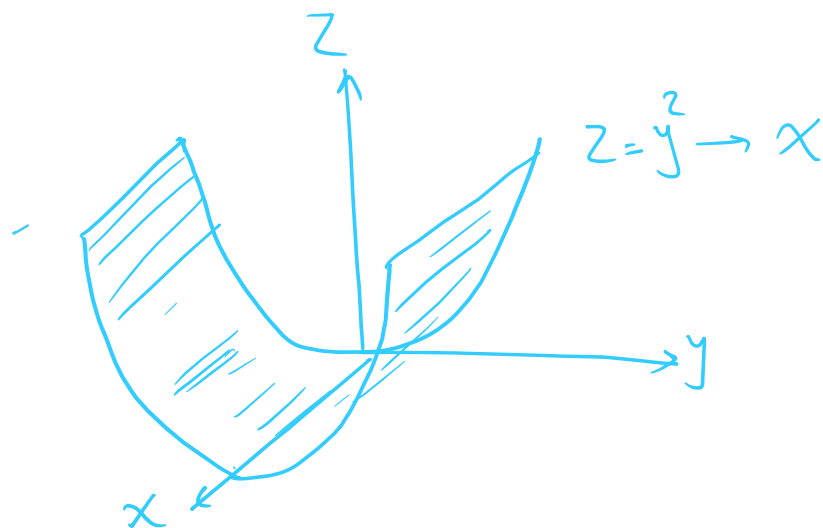
مثال: معادله رویه حاصل از دوران هندولس $\frac{x^2}{9} - \frac{y^2}{4} = 1$ حول محور x بیاید.



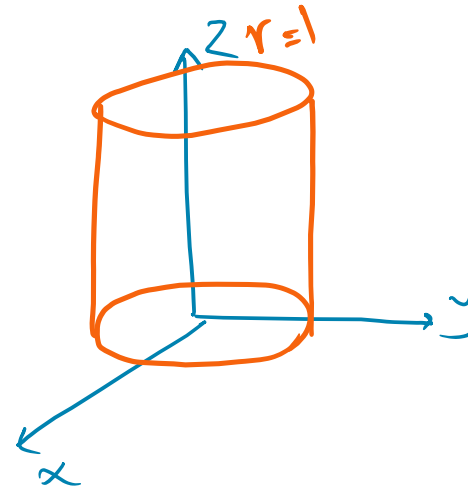
$$\left. \begin{array}{l} x \rightarrow x \\ y \rightarrow \sqrt{y^2 + z^2} \end{array} \right\} \Rightarrow \frac{x^2}{9} - \frac{(\sqrt{y^2 + z^2})^2}{4} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{4} = 1$$

2- استوانه

$$\mathbb{R}^3 \quad x, y, z \rightarrow \begin{array}{ccc} x & x & y \\ y & z & x \\ \hline & & z \end{array}$$



$$\underbrace{x^2 + y^2 = 1}_Z \Rightarrow \begin{cases} \mathbb{R}^2 \Rightarrow \text{دایره} \\ \mathbb{R}^3 \Rightarrow \text{استوانه} \end{cases}$$



3- رویه‌های درجه 2

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

1.3) بیضگون

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \xrightarrow{\times a^2 b^2 c^2} \underbrace{b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2}_{Ax^2 + By^2 + Cz^2 = k}$$

2-3 مخروط cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$x^2 = 4y^2 + z^2$$

$$\left\{ \begin{array}{l} x=1 \Rightarrow \underline{4y^2 + z^2 = 1} \Rightarrow \text{بيضا} \\ y=1 \quad x^2 = 4 + z^2 \Rightarrow x^2 - z^2 = 4 \quad \text{هذلولى} \\ z=1 \quad x^2 - 4y^2 = 1 \Rightarrow \text{هذلولى} \end{array} \right.$$

3-3 (3-3) سيميكون بيضا

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \quad \left\{ \begin{array}{l} x \text{ و } z \\ y \text{ و } z \end{array} \right. \text{صفات تقاطع}$$

مسألة

$$\underbrace{x^2 + 2z^2 - 6x - y + 10 = 0}$$

$$x^2 - 6x + 9 + 2z^2 = y - 1$$

$$(x - 3)^2 + 2z^2 = y - 1$$

نعم) $x = y^2 + 4z^2 \quad \begin{cases} x = k \\ y = k \\ z = k \end{cases}$

$$x^2 + y^2 - 2x - 6y - z + 10 = 0$$

3-4) سه گون هندولی

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c} \quad \text{صفحات تقارن} \quad \begin{cases} xoz \\ yoz \end{cases}$$

3-5) هندولسه گون یکپارچه

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{صفحات تقارن} \quad \begin{cases} xoz \\ yoz \end{cases}$$

3-6) هندولسه گون دوپارچه

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{صفحات تقارن} \quad \begin{cases} xoz \\ yoz \end{cases}$$

مثال) معادله روبه‌رو مدبره یک مخروط است. a را بیابید

$$x^2 + 4y^2 - 6x - z^2 + 2z = a$$

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$a = ?$

$$4y^2 + x^2 - 6x + 9(-9) + z^2 + 2z + 1(-1) = a$$

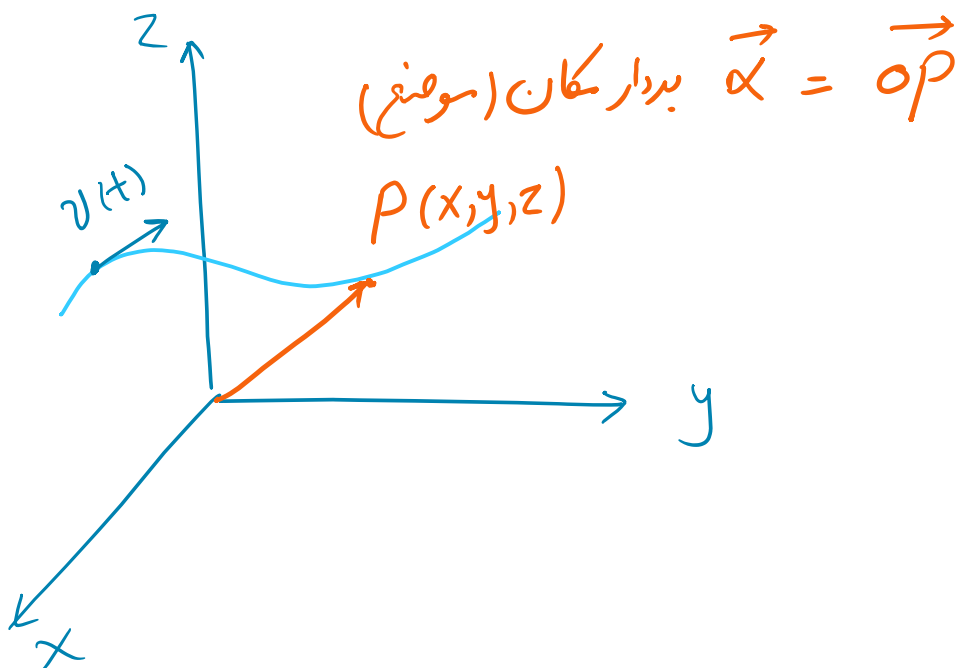
$$(x-3)^2 - (z-1)^2 + 4y^2 = a \Rightarrow (x-3)^2 - (z-1)^2 + 4y^2 = \underbrace{a+8}_0 \Rightarrow a+8=0$$

$$\boxed{a = -8}$$

خم پارامتره

رابطه در فضای دو یا چند بعدی که یک (یا چند) متغیر را گرفته و فرض بر آن یک بردار است.

$$\vec{\alpha}(t) = (x(t), y(t), z(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad \mathbb{R} \rightarrow \mathbb{R}^3$$



بردار سرعت
(محاسن بردار سرعت)

$$\vec{v}(t) = \vec{\alpha}'(t) = (x'(t), y'(t), z'(t))$$

تند

$$|\vec{v}(t)| = \sqrt{x'^2 + y'^2 + z'^2}$$

$$\vec{a}(t) = \vec{v}'(t) = (x''(t), y''(t), z''(t))$$



بیدار قائم واحد

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

بیدار قائم واحد (دم)

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

طول قوس $S = \int_a^b |\vec{v}| dt$

$$ds = |\vec{v}| dt = \sqrt{x'^2 + y'^2 + z'^2} dt$$

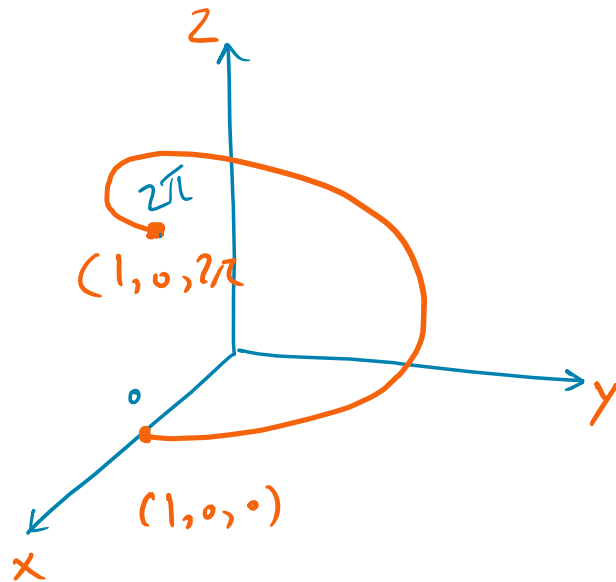
$$r(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k} \quad \text{بردار مکان}$$

طول قوس را از نقطه $(1, 0, 0)$ تا $(1, 0, 2\pi)$ به دست آورید
 $t=0$ $t=2\pi$

$|v|$
 $\downarrow$
 $\alpha(t) \hookrightarrow r'(t) \Rightarrow r'(t) = -\sin t \vec{i} + \cos t \vec{j} + \vec{k} \Rightarrow$
 $|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_0^{2\pi} \sqrt{2} dt = \sqrt{2} t \Big|_0^{2\pi}$$

$$= \underline{2\sqrt{2}\pi}$$



* تمرین: اگر از نقطه $(1, 0, 0)$ شروع کنیم و به اندازه $2\sqrt{2}\pi$ جلو برویم. لان در کجا میسیم

$$\sqrt{2}t = 2\sqrt{2}\pi \Rightarrow t = 2\pi$$

مثال: معادله صفحه عمود بر منحنی در نقطه $t = \frac{\pi}{4}$ محاسبه شود.

$$\begin{cases} x = \sin t \\ y = \sin t \\ z = \cos 2t \end{cases}$$

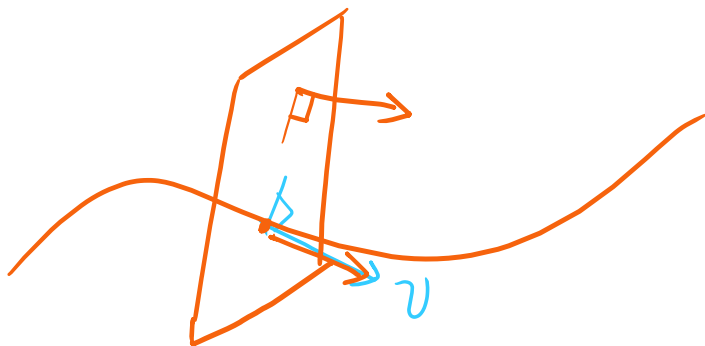
$$\alpha(t) = \langle \sin t, \sin t, \cos 2t \rangle$$

$$P_0 \Rightarrow \alpha\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$v(t) = \alpha'(t) = \langle \cos t, \cos t, -2\sin 2t \rangle \quad v\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -2\right)$$

مختص

نرمال صفحه



$$\Rightarrow \frac{\sqrt{2}}{2} \left(x - \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} \left(y - \frac{\sqrt{2}}{2}\right) - 2(z - 0) = 0$$

معادله صفحه

$$C \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases} \quad P_6 \Rightarrow \underline{t=1}$$

نقطه آویز منحنی در P_6

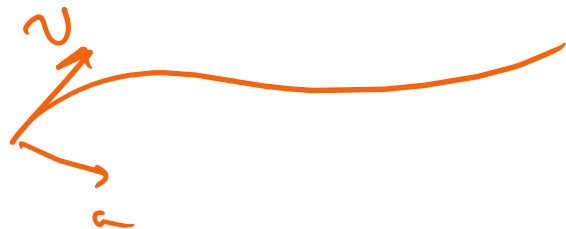
$$\alpha(t) = (t, t^2, t^3) \Rightarrow \alpha(1) = (1, 1, 1)$$

$$v = \alpha'(t) = (1, 2t, 3t^2) \Rightarrow v(1) = (1, 2, 3)$$

$$a = v'(t) = (0, 2, 6t) \Rightarrow a(1) = (0, 2, 6)$$

$$\left. \begin{matrix} v \times a \end{matrix} \right\} \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} \Rightarrow (6, -6, 2)$$

میدار نرمال منحنی



معادله منحنی برشمار
 $\Rightarrow$

$$6(x-1) - 6(y-1) + 2(z-1) = 0$$

* $N(t)$ بردار یکه قائم به مسیر

$$r(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$$

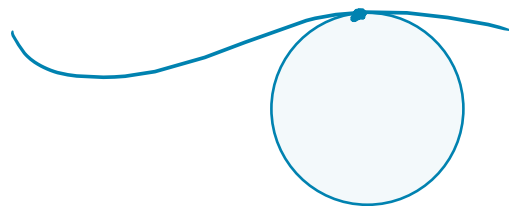
$-\cos t \vec{i} - \sin t \vec{j}$ یکایک ۸

* انحنای
انحنای یک منحنی در یک نقطه: سرعت در خشت ضلعها من به منحنی

$$k(t) = \frac{|T'(t)|}{|r'(t)|}$$

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$k(t) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$



$\infty = \text{شعاع انحنای} \Rightarrow \infty = \text{انحنای}$

$$R = \frac{1}{k}$$

* مثال) مختصر $r(t) = \langle t, t^2, t^3 \rangle$ را در نقطه $(0, 0, 0)$ بدست آورید.

$$r'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$|r'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$t = 0$$

$$r''(t) = \langle 0, 2, 6t \rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \underbrace{(12t^2 - 6t^3)}_{6t^2} i - (6t - 0) j + (2 - 0) k = 6t^2 i - 6t j + 2 k$$

$$|r'(t) \times r''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$k(t) = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}} = 2$$

$$\left. \begin{array}{l} x^2 = 2y \text{ استوانه سهمی} \\ 3z = xy \end{array} \right\} \begin{array}{l} \text{طول دقیق منحنی حاصل از} \\ \text{برسردار از مبدأ تا} \\ (6, 18, 36) \end{array}$$

$$S = \int_a^b |v| dt$$

$$\text{مبدأ} \Rightarrow t=0 \quad p=(0,0,0)$$

$$P \rightarrow t=6 \quad p=(6,18,36)$$

$$S = \int_0^6 \left(1 + \frac{t^2}{2}\right) dt = t + \frac{t^3}{6} \Big|_0^6 = 6 + \frac{6^3}{6} = 6 + 36 = \underline{42}$$

$$x=t$$

$$r(t) = \left\langle t, \frac{t^2}{2}, \frac{t^3}{6} \right\rangle$$

$$r'(t) = \left\langle 1, t, \frac{t^2}{2} \right\rangle$$

$$|r'(t)| = \sqrt{1 + t^2 + \frac{t^4}{4}} = \sqrt{\left(1 + \frac{t^2}{2}\right)^2} = 1 + \frac{t^2}{2}$$

مثال) اگر $r'(t)$ به صورت زیر باشد آنگاه $r(t)$ را محاسبه کنید.

$$r'(t) = 2t \vec{i} + 3t^2 \vec{j} + \sqrt{t} \vec{k}$$

$$\underline{r(1) = \vec{i} + \vec{j}}$$

$$r(t) = \int r'(t) dt = \int (2t \vec{i} + 3t^2 \vec{j} + \sqrt{t} \vec{k}) dt$$

$$r(t) = t^2 \vec{i} + t^3 \vec{j} + \frac{2}{3} t^{3/2} \vec{k} + C$$

$$r(1) = \underbrace{\vec{i} + \vec{j} + \frac{2}{3} \vec{k}} + C$$

$$\frac{2}{3} \vec{k} + C = 0 \Rightarrow C = -\frac{2}{3} \vec{k}$$

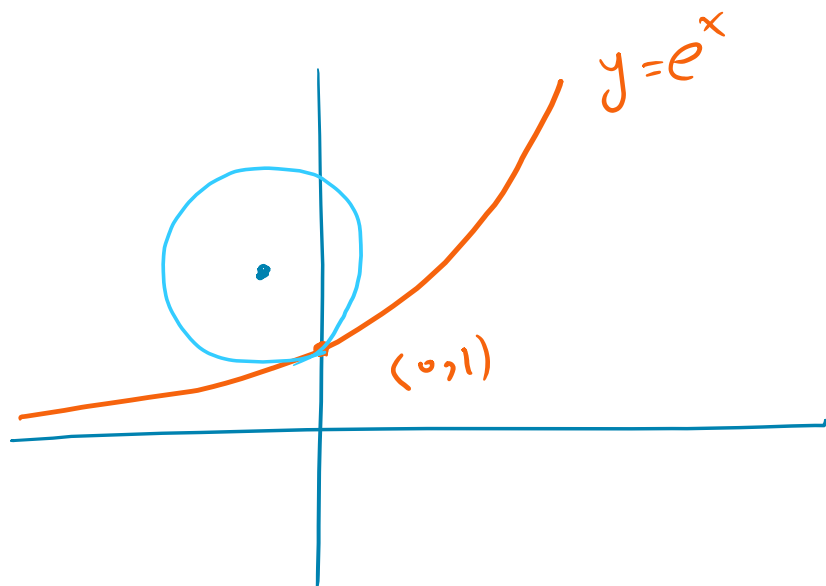
$$r(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$$

T, N, B

تمرین ~ میدان یک ماسه سوز

$$y = e^x$$

$$x = 0$$



$$y = e^x$$

α
 β
 R

$$K = \frac{|y''|}{(1+y'^2)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$$

$$x=0 \Rightarrow K = \frac{1}{(2)^{3/2}}$$

مثال:
دایره ایما به
سبب $y = e^x$ در نقطه
 $x=0$

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

$$r = \frac{1}{K} = 2^{3/2} \Rightarrow r^2 = 2^3 = 8$$

$$(x-\alpha)^2 + (y-\beta)^2 = 8 \Rightarrow (x-\alpha)^2 + (y-\beta)^2 = 8 \Rightarrow \alpha = -2, \beta = 3$$

$$\Rightarrow \begin{cases} f(x) = g(x) \\ f'(x) = g'(x) \end{cases}$$

$$\Rightarrow (0-\alpha)^2 + (1-\beta)^2 = 8 \Rightarrow \boxed{x^2 + (1-\beta)^2 = 8} \quad \text{مقابل 1}$$

$$y = e^x \Rightarrow e^0 = 1 \Rightarrow y' = \frac{F'_x}{F'_y} = \frac{2(x-\alpha)}{2(y-\beta)} = 1 \Rightarrow \frac{\alpha}{1-\beta} = 1 \Rightarrow \boxed{\alpha = 1-\beta} \quad \text{مقابل 2}$$

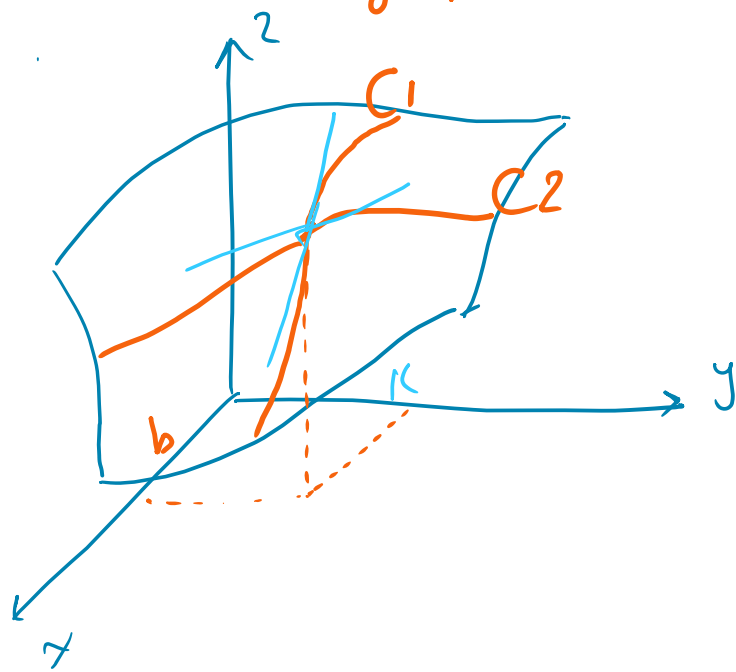
مشتقات جزئی

$$z = f(x, y)$$

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x = \frac{\partial f}{\partial x} = \frac{\delta}{\delta x} f(x, y) = f_1 = D_1 f$$

$$f_y(x, y) = \dots$$

$y = k \Rightarrow$ منحنی



$$z = f(x, y)$$

$$f_x(a, b) = \bigcirc \begin{matrix} \text{شیب خط مماس به} \\ \text{منحنی } C_1 \end{matrix} \quad \begin{cases} y = k \\ z = f(x, k) \end{cases}$$

$$f_y(a, b) = \bigcirc \begin{matrix} \text{شیب خط مماس به منحنی} \\ C_2 \end{matrix} \quad \begin{cases} x = b \\ z = f(b, y) \end{cases}$$

$$f(x,y) = xy^2 - x^3y \quad f_x = ? \quad y^2 - 3yx^2$$

$$f_y = 2xy - x^3$$

$$z = \ln(x+t^2) \quad z_t, z_x = ?$$

$$y = \ln u \Rightarrow y' = \frac{u'}{u}$$

$$z_t = \frac{\partial z}{\partial t} = \frac{2t}{x+t^2}$$

$$z_x = \frac{\partial z}{\partial x} = \frac{1}{x+t^2}$$

$$f(x,y,z) = e^{xy} \ln z \quad f_x, f_y, f_z = ?$$

$$x^2 + 2y^2 + 3z^2 = 1 \Rightarrow \frac{\partial}{\partial x} (x^2 + 2y^2 + 3z^2) = \frac{\partial}{\partial x} (1)$$

$$2x = ? \quad \frac{\partial z}{\partial x}$$

$$2x + 0 + 6z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-x}{3z}$$

عمرين) $e^2 = xyz \quad \frac{\partial z}{\partial x}$

$$\frac{\partial}{\partial x} (e^2) = \frac{\partial}{\partial x} (xyz)$$

$$\frac{\partial z}{\partial x} = \frac{yz}{e^2 - xy}$$

$$e^2 \frac{\partial z}{\partial x} = yz + \frac{\partial z}{\partial x} \cdot xy \Rightarrow e^2 \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial x} = yz$$

$$\frac{\partial z}{\partial x} (e^2 - xy) = yz \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^2 - xy}}$$

مثال) $f(x, y) = 16 - 4x^2 - y^2$ $P(1, 2, 8)$ سطح مماس

مثال) $Z = 2x^2 + y^2$ $P(1, 1, 3)$ معادله صفحه مماس $Z - Z_0 = f_x(x - x_0) + f_y(y - y_0)$

$$f_x = 4x \Rightarrow f_x = 4$$

$$f_y = 2y \Rightarrow f_y = 2$$

$$Z - 3 = 4(x - 1) + 2(y - 1)$$

مستقيمات مماس بالانحراف

$$Z = f(x, y)$$

f_{xx}
 f_{xy}
 f_{yx}
 f_{yy}

مستقيمات مماس (درم)

① ②

$$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{11} = (f_x)_x$$

$$f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y$$

① ②

$$f(x,y) = x^4 y - 2x^3 y^2$$

$$\begin{array}{l} f_{xx} \\ f_{xy} \\ f_{yx} \\ f_{yy} \end{array}$$

نکته: اگر f در (x,y) پیوسته باشد،
 $f_{xy} = f_{yx}$

مثال) $f(x,y,z) = \sin(3x+yz)$

$$\begin{array}{c} f_{xyz} \\ \swarrow \downarrow \searrow \\ \textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4} \end{array}$$

$$f_x = 3 \cos(3x+yz) \xrightarrow{f_{xx}} -9 \sin(3x+yz) \xrightarrow{f_{xxz}} \underbrace{-9z \cos(3x+yz)}$$

$$\underline{f_{xxyz} \rightarrow -9 \cos(3x+yz) + 9zy \sin(3x+yz)}$$

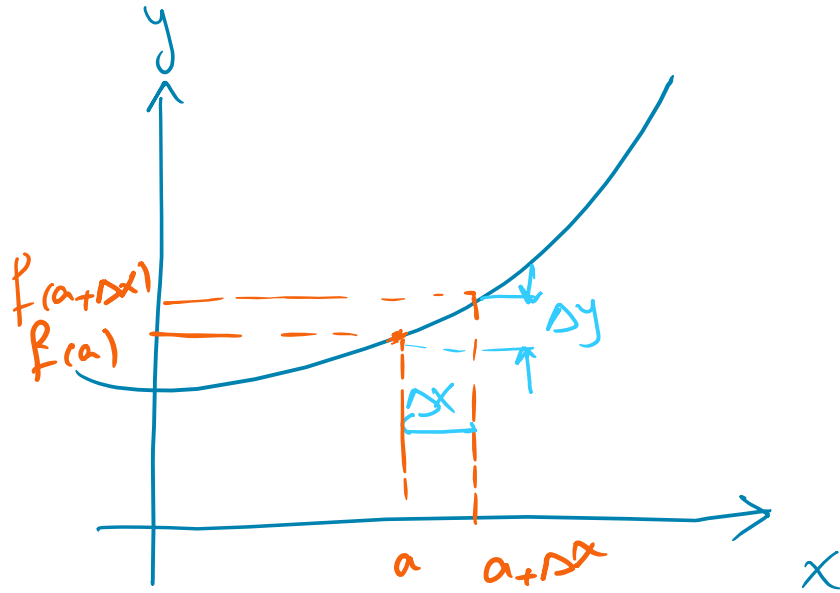
(تمرین)

$$w = \frac{x}{y+2z}$$

$$\frac{\partial^3 w}{\partial z \partial y \partial x}$$

(3) (2) (1)

$$w_{xyz} = \frac{4}{(y+2z)^3}$$



$$y = f(x) \Rightarrow dy = y' dx$$

$$\Delta y = f(a + \Delta x) - f(a)$$

تغییرات بی‌انتهی
دifferential

مثال) $y = x^2 - x + 1$

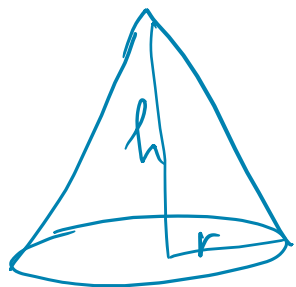
$$x=2, \Delta x=1 \quad \begin{cases} \Delta y \\ dy \end{cases}$$

$$dy = (2x-1)dx \Rightarrow dy = (2 \times 2 - 1)(1) = 3$$

$$\Delta y = f(a + \Delta x) - f(a) = f(2+1) - f(2) = 7 - 3 = 4$$

$$z = f(x, y)$$

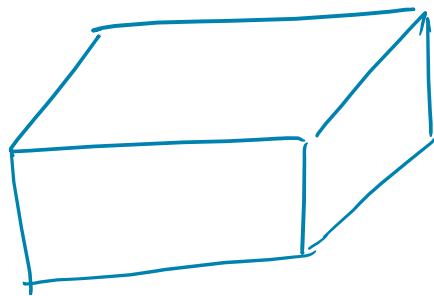
$$dz = \frac{\partial z}{\partial x} \times dx + \frac{\partial z}{\partial y} \times dy$$



$$v = \frac{\pi r^2 h}{3}$$

$$dv = \frac{\partial v}{\partial r} \times dr + \frac{\partial v}{\partial h} \times dh$$

$$dv = \frac{2}{3} \pi h r \times dr + \frac{\pi r^2}{3} dh$$



$$v = xyz$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$dv = yz dx + xz dy + xy dz$$

$$dv = 3 \times 4 \times 0.1 + 2 \times 4 \times 0.1 + 2 \times 3 \times 0.1 = 2.6$$

$$\Delta v = f(2.1, 3.1, 4.1) - f(2, 3, 4) = 2.6$$

$$\begin{matrix} x=2, & y=3, & z=4 \\ 2.1 & 3.1 & 4.1 \end{matrix}$$

$$\underbrace{\sin x}_{x=0} \approx \underbrace{x}$$

$$z = f(x, y)$$

$$\underline{l(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

ظرياً

مثال) $f(x, y) = x e^{xy}$ $(1, 0)$

$$l(x, y) = f(1, 0) + f_x(x - 1) + f_y(y - 0) = 1 \times e^0 + \underbrace{(e^{xy} + xy e^{xy})}_{1 \times (x-1)}(x - 1) + \frac{x^2 e^{xy}}{1}(y - 0) = x + y$$

$$x e^{xy} \approx x + y \quad (1, 0)$$

تمرين) $f(2, 5) = 6$ $f_x(2, 5) = 1$ $f_y(2, 5) = -1$ $f(2.2, 4.9) = ?$

6,3

$$l(x, y) = \underbrace{f(a, b)}_{6} + \underbrace{f_x}_{1}(x - a) + f_y(y - b)$$

$$l(x, y) = 6 + 1 \times (x - 2) + (-1)(y - 5) = x - y + 9 \Rightarrow$$

قاعده زنجیره ای

$$Z = f(x, y)$$

$$x = g(t)$$

$$y = h(t)$$

$$\frac{\partial Z}{\partial t} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

مثال) $Z = x^2 y + 3xy^4$

$$x = \sin 2t$$

$$y = \cos t$$

$$\frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x} \times \frac{dx}{dt} + \frac{\partial Z}{\partial y} \times \frac{dy}{dt}$$

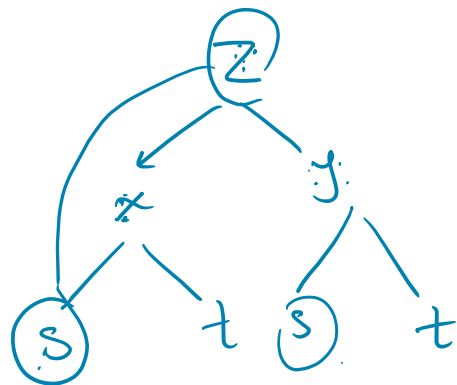
$$\frac{\partial Z}{\partial t} = (2xy + 3y^4) \times (2\cos 2t) + (x^2 + 12xy^3) \times (-\sin t)$$

$$\left. \frac{\partial Z}{\partial t} \right|_{t=0} = 3 \times 2 + 0 = 6$$

$$z = f(x, y)$$

$$x = g(s, t)$$

$$y = h(s, t)$$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial t}$$

Ex) $z = e^x \sin y$

$$x = st^2$$

$$y = s^2 t$$

$$\frac{\partial z}{\partial s} = e^x \sin y \times t^2 + e^x \cos y \times 2st$$

Ex) $g(s, t) = f(\underbrace{s^2 - t^2}_x, \underbrace{t^2 - s^2}_y)$ $\frac{\partial g}{\partial s} = ?$

$$x = s^2 - t^2$$

$$y = t^2 - s^2$$

$$\frac{\partial g}{\partial s} = \left(\frac{\partial f}{\partial x} \right) \times \frac{\partial x}{\partial s} + \left(\frac{\partial f}{\partial y} \right) \times \frac{\partial y}{\partial s}$$

$\downarrow$ $2s$
 $\downarrow$ $-2s$

$$F(x, y) = x^2 + y^3 - 6xy$$

$$\frac{dy}{dx} = - \frac{F_x}{F_y} = - \frac{2x - 6y}{3y^2 - 6x}$$

$$\frac{\partial F}{\partial x}$$

$$- \frac{\partial F}{\partial y}$$

مستقيم

مثال) $y \cos x = x^2 + y^2$

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$

$$F = y \cos x - x^2 - y^2$$

$$\frac{dy}{dx} = - \frac{-y \sin x - 2x}{\cos x - 2y} = \frac{2x + y \sin x}{\cos x - 2y}$$

تمرين) $x^3 + y^3 + z^3 + 6xyz = 1$

$$\frac{\partial z}{\partial x} ? \quad \frac{\partial z}{\partial y} ?$$

مسئله جهت یابی

$$D_u f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$u = (a, b) \rightarrow$$

$$a\vec{i} + b\vec{j}$$

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$$

جهت یابی

$$\nabla f = \langle f_x, f_y \rangle$$

$$f_x \vec{i} + f_y \vec{j}$$

$$D_u f(x, y) = f_x(x, y) \cdot a + f_y(x, y) \cdot b$$

$$\text{مثال) } f(x, y) = x^2 y^3 - 4y \quad P(2, -1)$$

$$u = 2\vec{i} + 5\vec{j}$$

$$\nabla f(x, y) = f_x \vec{i} + f_y \vec{j} = 2y^3 x \vec{i} + (3x^2 y^2 - 4) \vec{j}$$

$$D_u f(2, -1) = ? = \frac{2(-1)^3(2) \times \frac{2}{\sqrt{29}} + (3(2)^2(-1)^2 - 4) \frac{5}{\sqrt{29}}}{|u| = \sqrt{4 + 25} = \sqrt{29}} \quad u = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

مسئله جهت یابی

$$D_u f(x, y) = \nabla f(x, y) \cdot \underline{u}$$

مثال) $f(x, y) = x e^y$

$M(2, 0) \quad P(2, 0) \quad Q(\frac{1}{2}, 2)$

$$\vec{v} = \vec{PQ} = \langle -\frac{3}{2}, 2 \rangle \quad -\frac{3}{2}\vec{i} + 2\vec{j}$$

$$|\vec{v}| = \sqrt{(-\frac{3}{2})^2 + (2)^2} \Rightarrow \underline{u = \langle -\frac{3}{5}, \frac{4}{5} \rangle}$$

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$\downarrow \quad \downarrow$$

$$e^y \quad x e^y$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

$$D_f(2, 0) = \nabla f \cdot u = 1 \times (-\frac{3}{5}) + 2 \times \frac{4}{5} = 1$$

* نکته: ما کم ترین تغییرات در مقدار است و درجه جهت است.

$$D_u f = \nabla f \cdot u = |\nabla f| \cdot \widehat{|\underline{u}|} \cdot \widehat{C, \theta} = |\nabla f|$$

$$f(x,y) = x^3 - 3xy + 4y^2$$

$$\theta = \frac{\pi}{6}$$

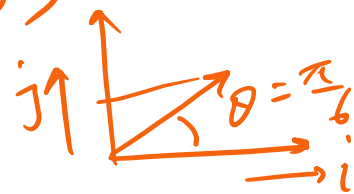
$$D_u f(1,2) = ?$$

$$u = \left(\frac{\sqrt{3}}{2} \vec{i} + \frac{1}{2} \vec{j} \right) \quad \theta =$$

$$u = \left\langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 3y, -3x + 8y \rangle \quad \nabla f(1,2) = \langle -3, 13 \rangle$$

$$\underline{D_u f} = \nabla f \cdot u = -3 \times \frac{\sqrt{3}}{2} + 13 \times \frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}$$



$$D_u f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h a, y_0 + h b, z_0 + h c) - f(x_0, y_0, z_0)}{h}$$

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot u \quad \nabla f = \langle f_x, f_y, f_z \rangle$$

مثال) $f(x,y,z) = x^2 y + y^2 z \quad P(1,2,3) \quad v = \langle 2, -1, 2 \rangle$

بإذن $D_v f(1,2,3) = 1$

$$4 \times \frac{2}{3} + 13 \left(-\frac{1}{3} \right) + 4 \times \frac{2}{3}$$

معادله صفحه مماس بدیده

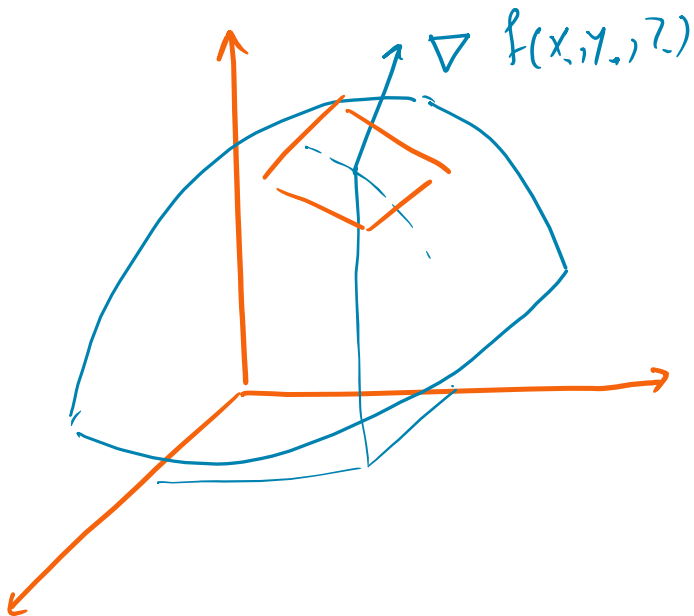
$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

مثال) $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \quad (-2, 1, -3)$

$$\begin{cases} F_x = \frac{x}{2} \\ F_y = 2y \\ F_z = \frac{2z}{9} \end{cases} \Rightarrow \begin{cases} F_x = -1 \\ F_y = 2 \\ F_z = -\frac{2}{3} \end{cases}$$

$$-1(x+2) + 2(y-1) + \left(-\frac{2}{3}\right)(z+3) = 0$$

$$3x - 6y - 2z + 18 = 0$$



(تمرین)

$$\begin{cases} z = x^2 + y^2 \\ 4x^2 + y^2 + z^2 = 9 \end{cases}$$

معادله خط مماس به منحنی است در نقطه $(-1, 1, 2)$

$$\begin{aligned} f(x, y, z) &= z - x^2 - y^2 \\ g(x, y, z) &= 4x^2 + y^2 + z^2 - 9 \end{aligned} \Rightarrow \begin{cases} \nabla f = \langle -2x, -2y, 1 \rangle \\ \nabla g = \langle 8x, 2y, 2z \rangle \end{cases} \Rightarrow \begin{cases} \nabla f = \langle 2, -2, 1 \rangle \\ \nabla g = \langle -8, 2, 4 \rangle \end{cases}$$

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = \underbrace{-10\vec{i} - 16\vec{j} - 12\vec{k}}_{\Rightarrow} \Rightarrow \frac{x - (-1)}{-10} = \frac{y - 1}{-16} = \frac{z - 2}{-12}$$

$$f(x, y) = \dots\dots\dots$$

$$\begin{matrix} f_{xx} & f_{xy} & f_{yx} & f_{yy} \end{matrix}$$

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow D = D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$$\underbrace{D > 0}_{\text{نقطهٔ ریز}} \begin{cases} f_{xx} > 0 & \text{نقطهٔ min} = (a, b) \\ f_{xx} < 0 & \text{نقطهٔ max} \end{cases}$$

$$D < 0 \Rightarrow (a, b) \text{ نه min و نه max است}$$

$$D = 0 \Rightarrow \text{از این زمین نتیجه ای نمی توان گرفت}$$

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

$$f_x = 2x - 2 = 0 \Rightarrow x = 1$$

(1, 3) نقطه بحر

$$f_y = 2y - 6 = 0 \Rightarrow y = 3$$

$$f_{xx} = 2 > 0$$

$$f_{xy} = 0$$

$$f_{yx} = 0$$

$$f_{yy} = 2$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \textcircled{4} > 0 \quad \text{و} \quad f_{xx} > 0$$

(1, 3) مینیمم محلی

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f_x = 4x^3 - 4y = 0 \quad x^3 - y = 0 \Rightarrow y = x^3$$

$$f_y = 4y^3 - 4x = 0 \quad y^3 - x = 0 \quad \leftarrow \underbrace{(x^3)^3 - x = 0}$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0 \quad \begin{cases} x=0 \\ x=-1 \\ x=1 \end{cases} \Rightarrow \begin{cases} (0,0) \\ (-1,1) \\ (1,-1) \end{cases}$$

$$\left. \begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= -4 \\ f_{yx} &= -4 \\ f_{yy} &= 12y^2 \end{aligned} \right\}$$

$$D = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

$$D = 144x^2y^2 - 16$$

$$\begin{cases} (0,0) \Rightarrow D = -16 < 0 \Rightarrow \\ (-1,1) \Rightarrow D = 128 > 0 \\ (1,-1) \Rightarrow D = 128 > 0 \end{cases}$$

نقطه زین

min محلی

min محلی

انتگرال دوگانه

$$\int f(x) dx \Rightarrow \underbrace{\int_c^d \int_a^b f(x,y) dy dx}$$

مثال) $\int_1^4 \int_0^2 (6x^2y - 2x) dy dx$

$$\int_0^2 (6x^2y - 2x) dy = 3x^2y^2 - 2xy \Big|_0^2 = 3x^2(4) - 4x - (0 - 0) = 12x^2 - 4x$$

$$\int_1^4 (12x^2 - 4x) dx = \frac{12x^3}{3} - \frac{4x^2}{2} \Big|_1^4 = \frac{12(4^3)}{3} - \frac{4(4)^2}{2} - \left[\frac{12}{3} - \frac{4}{2} \right] = 222$$

$$\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy dx$$

$$\int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) dy = x \ln y + \frac{1}{x} \times \frac{y^2}{2} \Big|_1^2 = x \cdot \ln(2) + \frac{2}{x} - \left[0 + \frac{1}{2x} \right] = \ln(2)x + \frac{3}{2x}$$

$$\int_1^4 \left(\ln 2 \cdot x + \frac{3}{2x} \right) dx = \ln(2) \cdot \frac{x^2}{2} + \frac{3}{2} \ln x \Big|_1^4 = \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4)$$

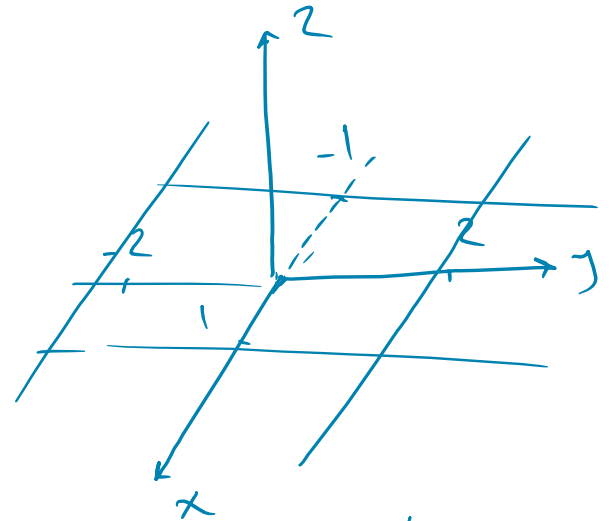
$$\text{flü}) \quad \frac{x^2}{4} + \frac{y^2}{9} + \textcircled{z} = 1 \quad R = \underbrace{[-1, 1]}_x \times \underbrace{[-2, 2]}_y$$

$$z = 1 - \frac{y^2}{9} - \frac{x^2}{4}$$

$$\int_{-2}^2 \int_{-1}^1 \left(1 - \frac{x^2}{4} - \frac{y^2}{9} \right) dx dy$$

$$\int_{-1}^1 \left(1 - \frac{x^2}{4} - \frac{y^2}{9} \right) dx = x - \frac{x^3}{12} - \frac{y^2}{9} x \Big|_{-1}^1 = \frac{11}{12} - \frac{y^2}{9}$$

$$\int_{-2}^2 \left(\frac{11}{12} - \frac{y^2}{9} \right) dy = \frac{11}{12} y - \frac{y^3}{27} \Big|_{-2}^2 = 4 \times \left(\frac{11}{12} \times 2 - \frac{8}{27} \right)$$



flü { } ~ ~ ~

$$\int_R \int y \sin(xy) dA = \int_0^\pi \int_1^2 y \sin(xy) dx dy$$

$$R = \underbrace{[1, 2]}_x \times \underbrace{[0, \pi]}_y$$

$$\int_1^2 y \sin(xy) dx = -\frac{\cos xy}{y} \cdot y \Big|_1^2 = -\cos 2y - (-\cos y)$$

$$\int_0^\pi (-\cos 2y + \cos y) dy = -\frac{1}{2} \sin 2y + \sin y \Big|_0^\pi = 0$$

$$\int_R \int g(x) \cdot h(y) dA = \int_a^b g(x) dx \times \int_c^d h(y) dy$$

$$R = [a, b] \times [c, d]$$

sub.

$$\checkmark \int_1^2 \int_0^3 \underbrace{x^2 y}_{\text{blue}} dx dy = \int_0^3 x^2 dx \times \int_1^2 y dy = \left. \frac{x^3}{3} \right|_0^3 \times \left. \frac{y^2}{2} \right|_1^2 = \frac{27}{2}$$

$$\int \int \left(\frac{x}{y} + \frac{y}{x} \right) dA$$

$$\int \int \sin x \cdot \cos y dA = \int_0^{\pi} \sin x dx \times \int_0^{\pi} \cos y dy = 1$$

$$R = \left[0, \frac{\pi}{2} \right] \times \left[0, \frac{\pi}{2} \right]$$

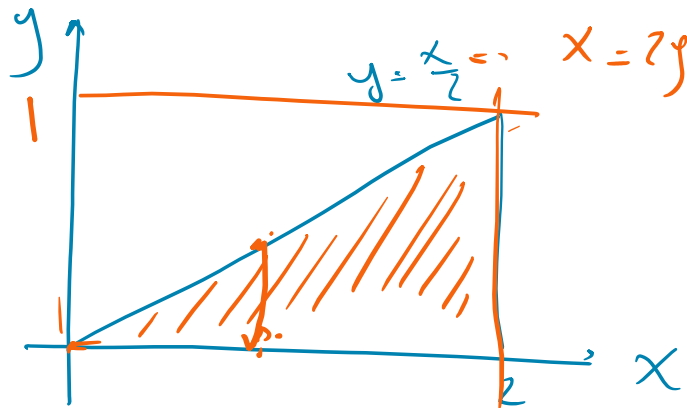
$$\int \int \frac{xy^2}{x^2+1} dA = \overbrace{\int_{-1}^1 \frac{x}{x^2+1} dx}^{\ln\left(\frac{x^2+1}{2}\right)} \times \int_{-3}^3 y^2 dy = \ln\left(\frac{x^2+1}{2}\right) \Big|_{-1}^1 \times \left. \frac{y^3}{3} \right|_{-3}^3 = 9 \ln 2$$

$$R = \{(x, y) \mid 0 \leq x \leq 1, -3 \leq y \leq 3\}$$

$$\int \int xy^2 dy dx$$

$$\int_0^2 \int_0^{\frac{x}{2}} xy^2 dy dx =$$

$$\int_0^1 \int_{2y}^2 xy^2 dx dy$$

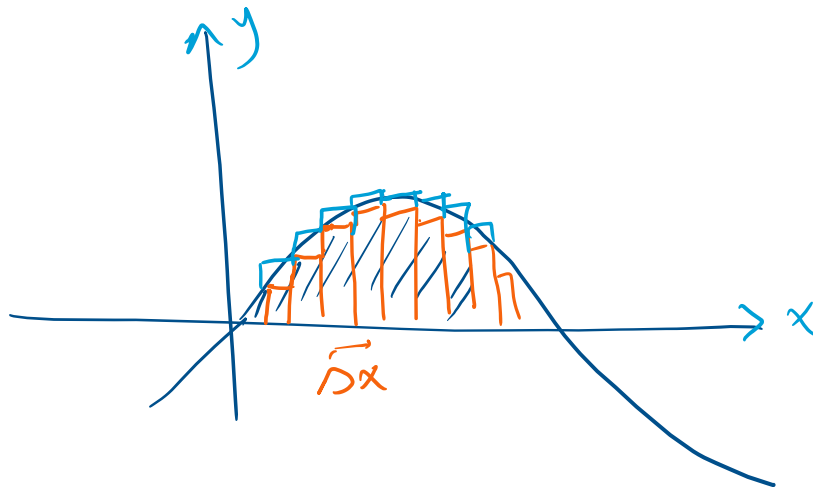


قانون متقارب

$$\int_0^{\pi} \sin x dx$$

x	sin x
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{4\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

$$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$$



$$\text{رمان پاسن} = (0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0) \times \frac{\pi}{6}$$

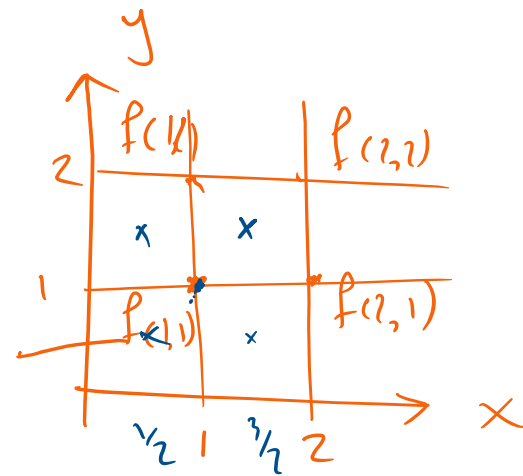
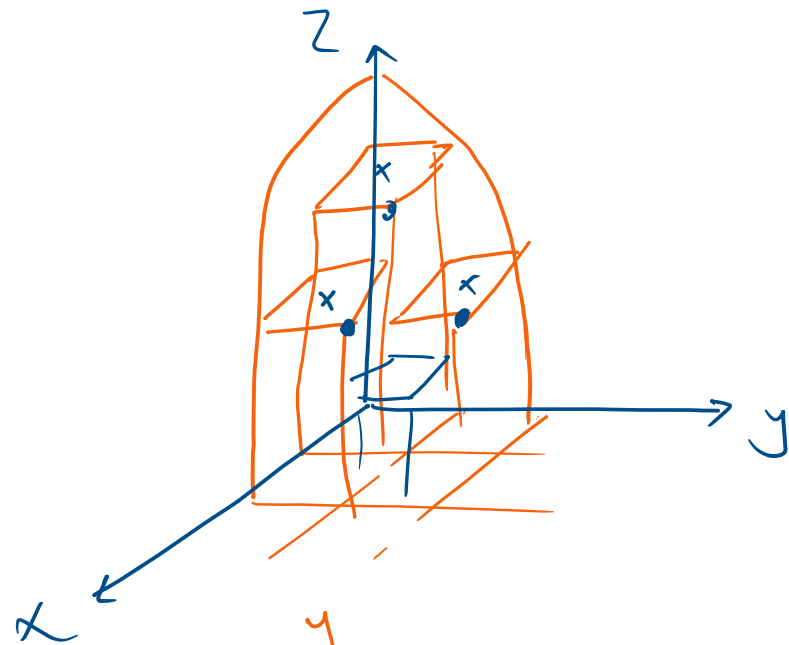
$$\text{غل} = (\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}) \times \frac{\pi}{6}$$

$$Z = 16 - x^2 - 2y^2$$

$$R = [0, 2] \times [0, 2]$$

$$\iint_R (16 - x^2 - 2y^2) dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) dx dy$$

$$\begin{aligned} \int \square &= [f(1,1) + f(1,2) + f(2,1) + f(2,2)] \times \Delta A \\ &= [13 + 1 + 10 + 4] \times 1 = 34 \end{aligned}$$

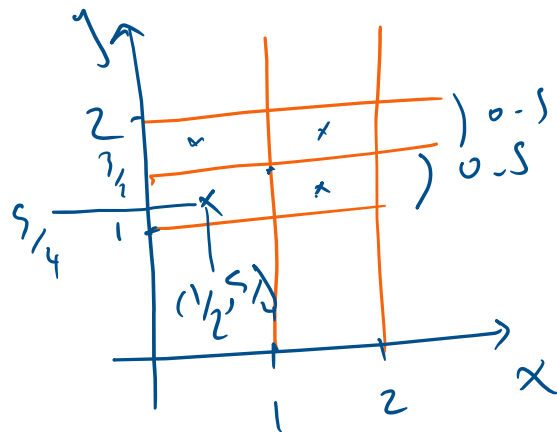


قانون نقطه‌باز

$$\iint_R (x - 3y^2) dA \rightarrow \begin{matrix} dx dy \\ dy dx \end{matrix}$$

$$R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

$$m = 2 \rightarrow \\ n = 2.$$



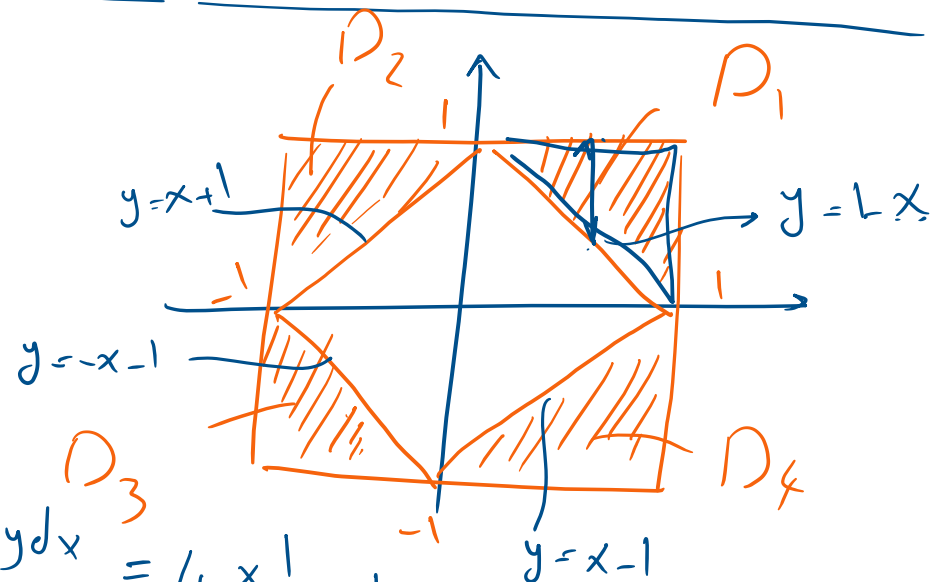
$$\left[f\left(\frac{1}{2}, \frac{5}{4}\right) + f\left(\frac{1}{2}, \frac{7}{4}\right) + f\left(\frac{3}{2}, \frac{5}{4}\right) + f\left(\frac{3}{2}, \frac{7}{4}\right) \right] \times \Delta A$$

Ex 8) $\iint_D x^2 dA$

$$\begin{matrix} D \\ (D_1 | D_2) \end{matrix}$$

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

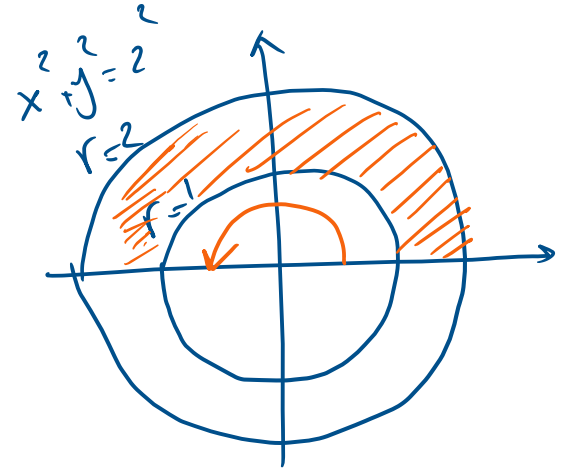
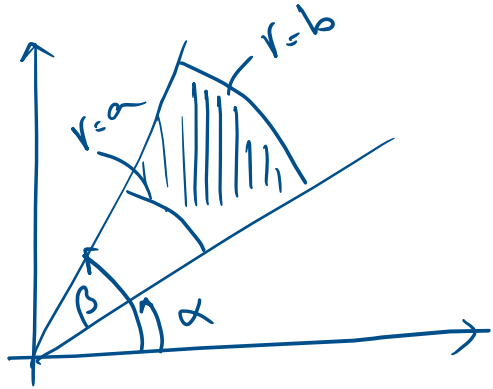
$$\int_0^1 \int_{1-x}^1 x^2 dy dx + \int_{-1}^0 \int_{x+1}^1 x^2 dy dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2 dy dx + \int_0^1 \int_{-1}^{x-1} x^2 dy dx = 4 \times \frac{1}{4} = 1$$



$$x = r \cos \theta \quad y = r \sin \theta \quad dx dy = \underline{r dr d\theta}$$

* انشکل درمانه در فضاات قطبی

$$\iint_R f(x,y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

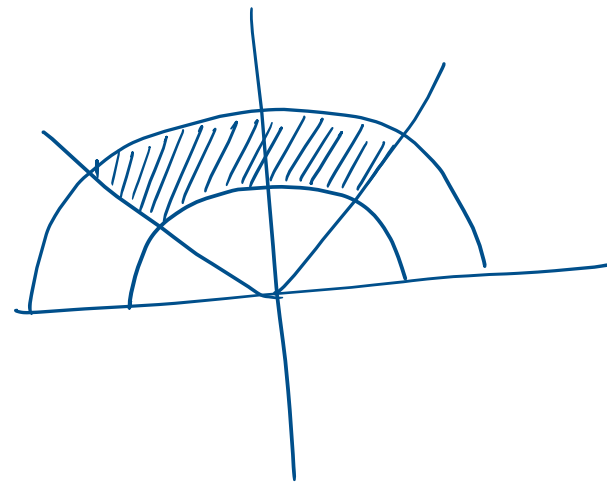


مثال) $\iint (3x + 4y^2) dA = \int_0^{\pi} \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta$

$$R \begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases} \quad \underline{y > 0} = \int_0^{\pi} \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta = \int_0^{\pi} (r^3 \cos \theta + r^4 \sin^2 \theta) \Big|_1^2 d\theta$$

$$= \int_0^{\pi} (7 \cos \theta + 15 \sin^2 \theta) d\theta = \left[7 \sin \theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \right]_0^{\pi} = \frac{15\pi}{2}$$

مثال) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^2 r dr d\theta = \frac{3\pi}{4}$



$$\iint_R g(x) h(y) dA = \int_a^b g(x) dx \int_c^d h(y) dy$$

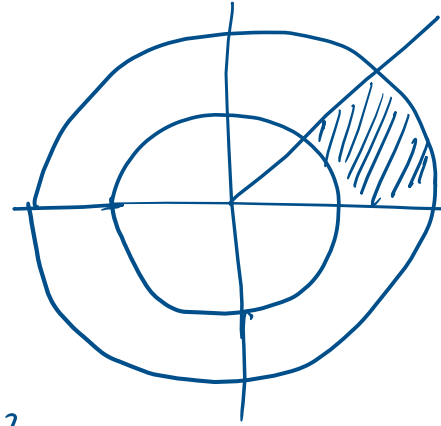
$$R = [a, b] \times [c, d]$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^2 r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \times \int_1^2 r dr = \theta \bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \times \frac{r^2}{2} \bigg|_1^2 = \frac{3\pi}{4}$$

مثال) $\int_0^{\frac{\pi}{2}} \int_0^2 e^{-r^2} dr d\theta = \frac{\pi}{4} (1 - e^{-4})$

تمرین) $\iint_R \arctan\left(\frac{y}{x}\right) dA =$

$$R = \{(x, y) \mid (x^2 + y^2 \leq 4) \cdot \{y \leq x\}\} \Rightarrow$$



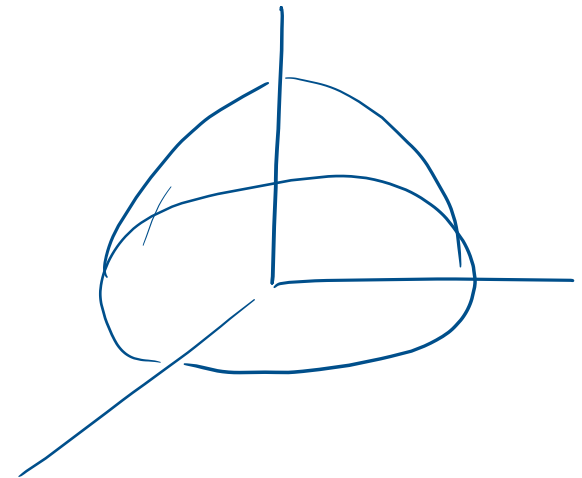
$$\iint (1 - x^2 - y^2) dA$$

$$z = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$$

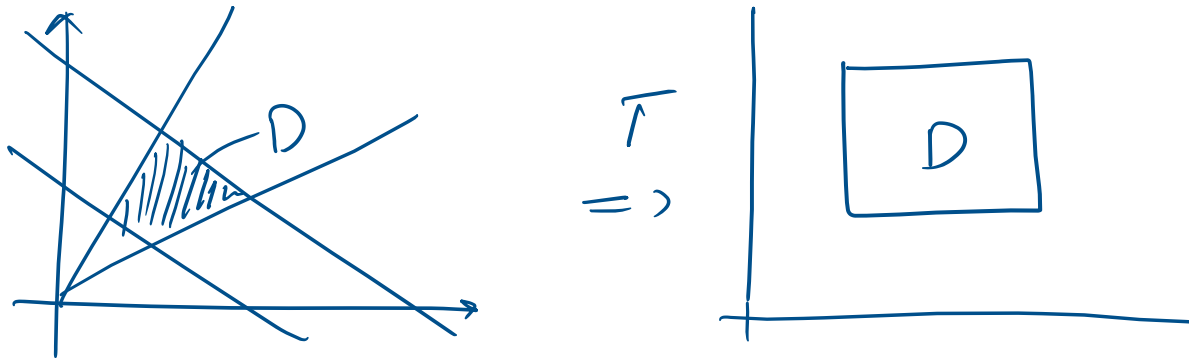
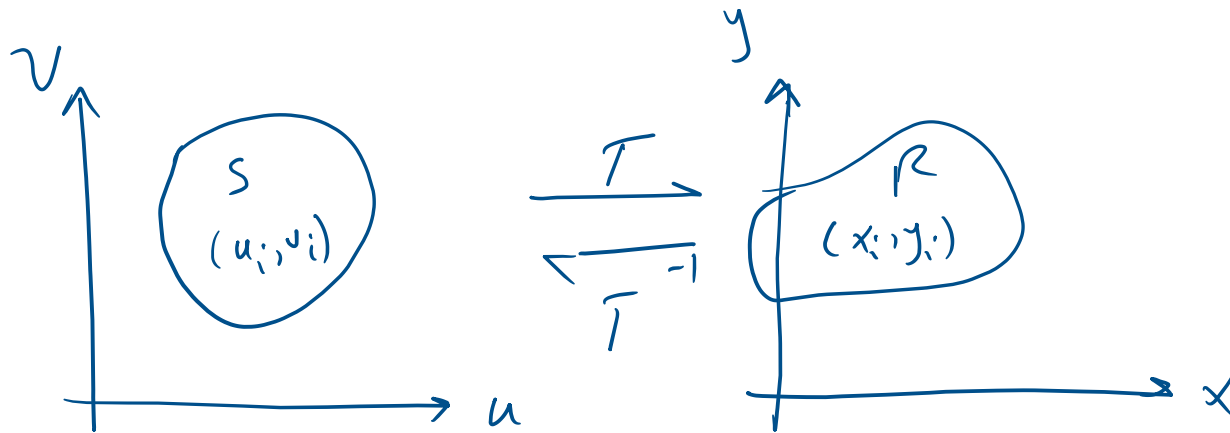
$z \geq 0$

$$\int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta = \int_0^{2\pi} d\theta \times \int_0^1 (r - r^3) dr$$

$$= 2\pi \times \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) \times \frac{\pi}{2}$$



تفسیر متغیر در انتگرال دوگانه



$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$J_f(x_1, \dots, x_n) = \frac{\partial(y_1, \dots, y_m)}{\partial(x_1, \dots, x_n)} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

10)

$$\iint_R \frac{x-2y}{3x-y} dA$$



$$\begin{cases} x-2y=0 \\ x-2y=4 \\ \underline{3x-y=1} \\ 3x-y=8 \end{cases}$$

$$x-2y=u \Rightarrow u=0$$

$$u=4$$

$$v=1$$

$$v=8$$

$$x-2y=u$$

$$3x-y=v$$

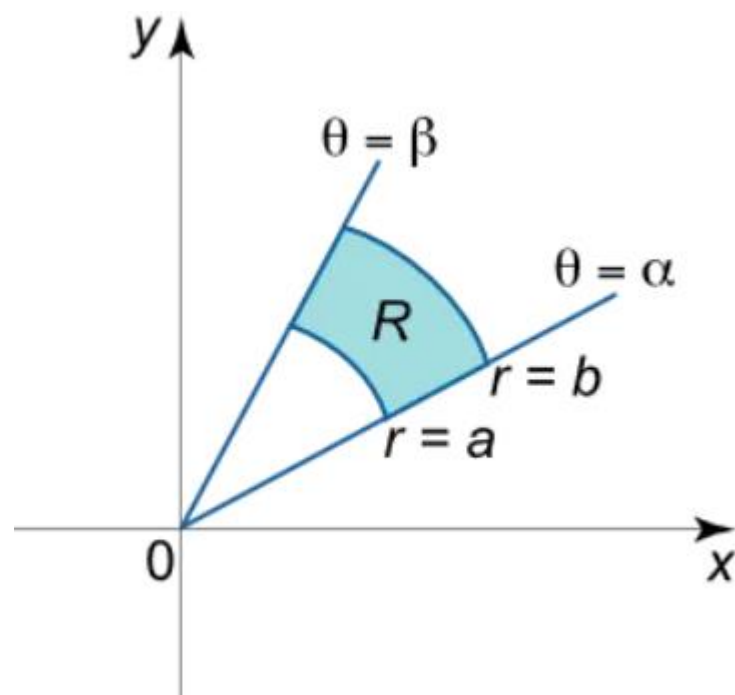


$$x = \frac{1}{5}(2v-u)$$

$$y = \frac{1}{5}(v-3u)$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}$$

$$\begin{aligned} \iint_R \frac{x-2y}{3x-y} dA &= \int_0^4 \int_1^8 \frac{u}{v} \times \frac{1}{5} \times dv du = \frac{1}{5} \int_0^4 u du \times \int_1^8 \frac{1}{v} dv = \frac{1}{5} \times \frac{u^2}{2} \Big|_0^4 \times \ln v \Big|_1^8 \\ &= \frac{8}{5} \ln 8 \end{aligned}$$



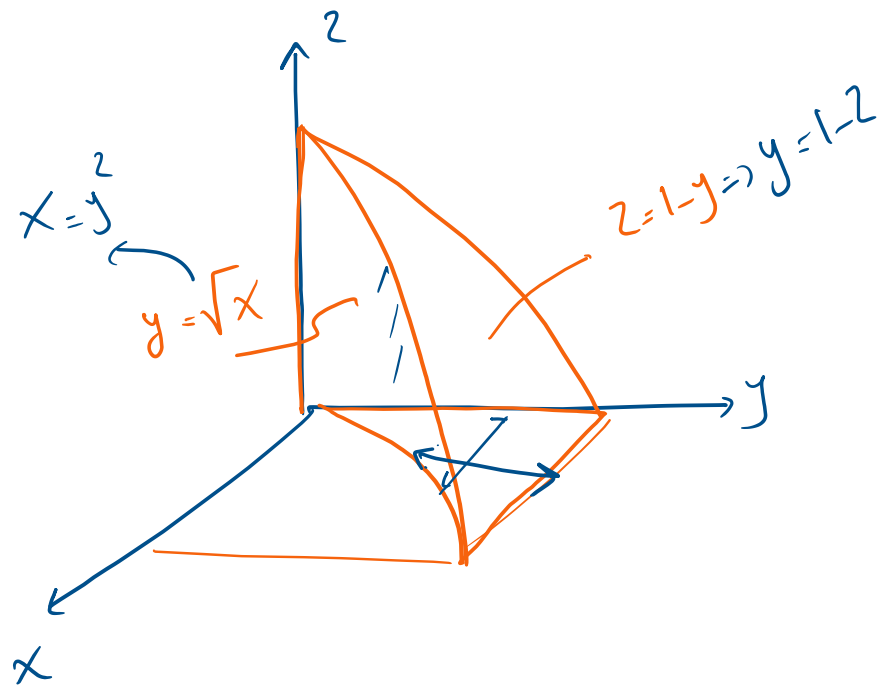
انتقال من 3D

$$\iiint_B f(x,y,z) dV = \int_r^s \int_c^d \int_a^b f(x,y,z) dx dy dz$$

$$B = \underbrace{[a,b]}_x \times \underbrace{[c,d]}_y \times \underbrace{[a,b]}_z$$

المسألة) $\iiint_B x y z^2 dV = \frac{27}{4}$

$$B = \{ (x,y,z) \mid 0 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq z \leq 3 \}$$



$$\Downarrow \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z) \frac{dz dy dx}{J_V}$$

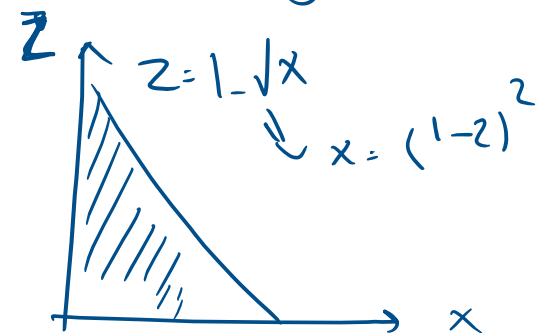
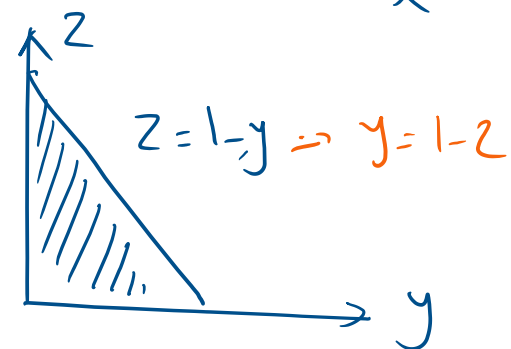
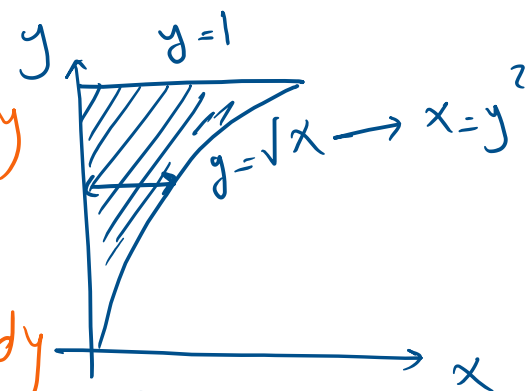
$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z) dz dx dy$$

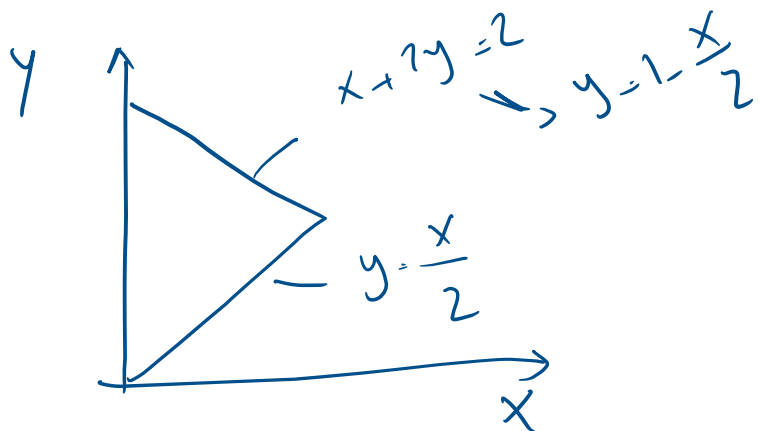
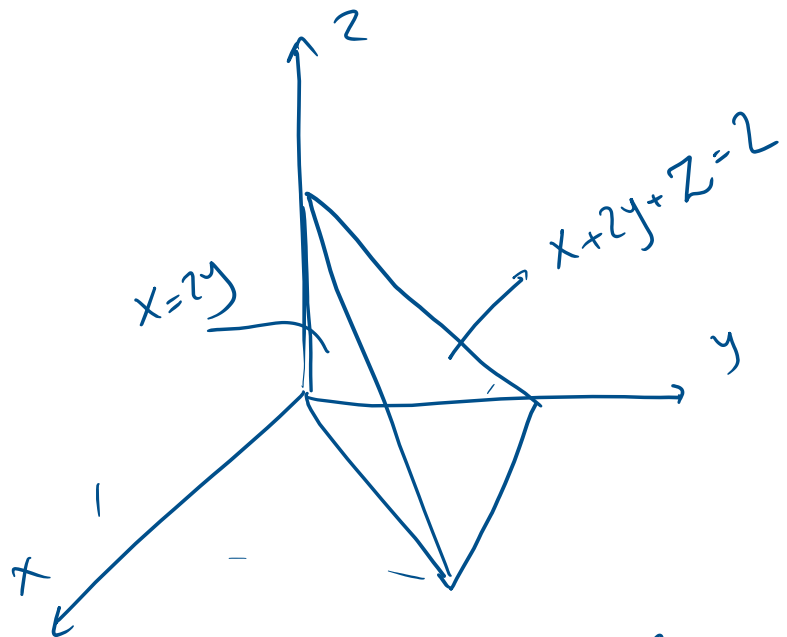
$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z) dx dz dy$$

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z) dx dy dz$$

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z) dy dx dz$$

$$\int_0^1 \int_{\sqrt{x}}^{1-z} \int_{\sqrt{x}}^{1-z} f(x,y,z) dy dz dx$$





$$\int_0^1 \int_{x/2}^{1-\frac{x}{2}} \int_0^{2-x-2y} dz dy dx$$

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